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Forgetting in multi-agent modal logics

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ABSTRACT

In the past decades, forgetting has been investigated for many logics and has found many applications in knowledge representation and reasoning. In this paper, we study forgetting in multi-agent modal logics. We adopt the semantic definition of existential bisimulation quantifiers as that of forgetting. We resort to canonical formulas of modal logics introduced by Moss. An arbitrary modal formula is equivalent to the disjunction of a unique set of satisfiable canonical formulas. We show that, for the logics of K_n, D_n, T_n, $K45_n$, $KD45_n$ and $S5_n$, the result of forgetting an atom from a satisfiable canonical formula can be computed by simply substituting the literals of the atom with $\top$. Thus we show that these logics are closed under forgetting, and hence have uniform interpolation. Finally, we generalize the above results to include common knowledge of propositional formulas.

1. Introduction

In the past decades, forgetting has been investigated for many logics and has found many applications in knowledge representation and reasoning (KR). Intuitively, forgetting some symbols from a theory should result in a theory that is weaker than the original theory and entails the same set of sentences that do not mention those symbols. We say that a logic is closed under forgetting if the result of forgetting can always be expressed in the same logic. A closely related concept is uniform interpolation. A logic has uniform interpolation if for any formula ϕ and any set S of symbols occurring in ϕ, there is a formula ψ, called a uniform interpolant of ϕ w.r.t. S, using only symbols in S such that ϕ and ψ entail the same set of formulas formulated only in S.

Over the years, forgetting in propositional logic has been used in abductive reasoning [30], reasoning under inconsistency [29], reasoning about knowledge [40], epistemic planning [22], etc. Forgetting in first-order logic has been used in computing first-order circumscription and identifying first-order frame conditions corresponding to modal axioms [15], progression for basic action theories in the situation calculus [32,33], and many other applications [16]. Forgetting in description logics has been applied to ontology reuse [44,42,25,36].

The seminal paper by Lin and Reiter [31] coined the name "forgetting". Nonetheless, the idea of forgetting in propositional logic dates back to Boole [5]. Forgetting in propositional logic is equivalent to propositional existential quantification: the result of forgetting atom p from formula ϕ is $\phi[p/\top] \vee \phi[p/\bot]$, where $\phi[p/\top]$ (resp. $\phi[p/\bot]$) denotes the result of replacing each occurrence of p in ϕ by *true* (resp. *false*). Besides propositional existential quantification, there are three other ways to compute forgetting in propositional logic. Firstly, take the conjunction of the finitely many non-equivalent

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logical consequences of ϕ which only mention atoms appearing in ϕ but do not mention p. Secondly, as a special case of the first method, convert ϕ to a conjunction of clauses, do all possible resolutions w.r.t. p, and take the conjunction of all clauses which do not mention p. Thirdly, convert ϕ to a disjunction of satisfiable terms where a *term* is a conjunction of literals, then substitute any occurrence of a literal p or $\neg p$ with $\top$, *i.e.*, substitute any occurrence of $\neg p$ with $\top$ and subsequently substitute any occurrence of p with $\top$ [28]. We call this method literal elimination. In the following, we will see that the above three methods have been generalized to other logics.

Lin and Reiter [31] gave a semantic definition of forgetting in first-order logic (FOL): the result of forgetting a predicate P in a theory T is a theory whose models are exactly those structures which agree with a model of T except possibly on the interpretation of P. They showed that for finite theories, forgetting a predicate coincides with second-order existential quantification, and thus is not first-order definable in general. Naturally, second-order quantifier elimination techniques can be used in computing the result of forgetting [15,16]. The notion of forgetting by Lin and Reiter was called strong forgetting by Zhang and Zhou [49], who proposed the notion of weak forgetting for FOL. A sentence ϕ is irrelevant to a predicate P if ϕ is logically equivalent to a sentence containing no occurrence of P. The result of weak forgetting a predicate P from a theory T is the set of first-order logical consequences of T irrelevant to P. They showed that weak forgetting is weaker than strong forgetting, and the two coincide when strong forgetting is first-order definable.

In recent years, forgetting has been generalized to various description logics (DLs) [44,42,25,36,50–52]. The community defines forgetting as the dual notion of uniform interpolation. There are different forms of forgetting including concept-forgetting, TBox-forgetting, ontology-forgetting, and query-forgetting. The research covers the following issues: whether a language is closed under forgetting, the complexity of deciding if a uniform interpolant exists, the size of a uniform interpolant, the complexity of computing forgetting, and methods for approximate forgetting. The majority of approaches for computing uniform interpolants are based on resolution [35,26,52], while ten Cate et al. [6] used the idea of literal elimination: to compute concept-forgetting for the expressive DL $\mathcal{ALC}$, they first transform a concept into a disjunctive form, and then do literal elimination.

Forgetting has also been studied for logic programming, in particular, for answer set programming (ASP). The non-monotonic nature of ASP creates unique challenges to forgetting in ASP. The community has proposed various concepts of forgetting satisfying different sets of postulates, including strong and weak forgetting by Zhang and Foo [47], semantic forgetting by Eiter and Wang [11], semantic strong and weak forgetting by Wong [46], knowledge forgetting by Wang et al. [43], etc. Gonçalves et al. [19] provided a uniform overview of existing forgetting operators and postulates in ASP.

Modal logics play an important role in AI. In a nutshell, AI is concerned with building intelligent agents. Agents possess mental attitudes, such as knowledge, beliefs, desires, goals, intentions, commitments, obligations, etc.; to reason about these mental attitudes and their dynamics, especially in the presence of multiple agents, one may fruitfully employ modal logics.

Forgetting for modal logics has also been investigated and applied to reasoning about knowledge and belief. Baral and Zhang [3] studied knowledge update, a special form of update with the effect that the agent becomes ignorant of a propositional formula. Van Ditmarsch et al. [10] presented a dynamic epistemic logic where the dynamic operator is the action of forgetting a propositional atom. Zhang and Zhou [48] studied forgetting in propositional S5 logic and its applications in knowledge updates and knowledge games. Liu and Wen [34] explored forgetting in first-order S5 logic and applied it to progression of knowledge in the situation calculus.

While forgetting in propositional logic is equivalent to propositional existential quantification, bisimulation quantifiers extend modal logics with a form of weak propositional quantification. They were introduced by Visser [41] and Ghilardi and Zawadowski [18] to semantically characterize uniform interpolants for modal logics. The intuitive idea is to quantify "modulo bisimulation": a model M satisfies a formula $\exists p\phi$ where p is an atom iff there is a model M' satisfying ϕ such that M and M' are bisimilar with exception on p. A logic is bisimulation invariant if any two bisimilar models satisfy the same set of formulas. It turns out that any logic that is invariant under bisimulation and closed under bisimulation quantification, that is, closed under elimination of the quantification, has uniform interpolation. However, the converse does not hold in general [9].

There has been considerable work on uniform interpolation in modal logics. It is well-known that K, T and S5 have uniform interpolation. Ghilardi [17] and Visser [41] gave constructive proofs that K has uniform interpolation. Bílková [4] showed that K and T have uniform interpolation by transforming a formula into a disjunctive form and then doing literal elimination. Wolter [45] showed that for S5, the uniform interpolant of a formula ϕ w.r.t. a subset P of atoms appearing in ϕ is the conjunction of the finitely many non-equivalent logical consequences of ϕ using only atoms from P. However, neither K4 nor S4 has uniform interpolation [4,18]. The reason is that when accessibility relations satisfy only transitivity, or only transitivity and reflexivity, formulas with bisimulation quantifiers can be used to express properties such as the existence of an infinite path where an atom has opposite truth values at neighboring states, which is not expressible using a quantifier-free formula. More concretely, in S4, $\exists p\exists q.[p \wedge \mathbf{K}(p \to \hat{\mathbf{K}}q) \wedge \mathbf{K}(q \to \hat{\mathbf{K}}p) \wedge \mathbf{K}(q \to r) \wedge \mathbf{K}(p \to \neg r)]$ is equivalent to the following infinite set of formulas: $\{\neg r, \hat{\mathbf{K}}r, \hat{\mathbf{K}}(r \wedge \hat{\mathbf{K}}\neg r), \hat{\mathbf{K}}(r \wedge \hat{\mathbf{K}}(\neg r \wedge \hat{\mathbf{K}}r)), \cdots\}$, which is not equivalent to any quantifier-free formula. Wolter [45] proved that uniform interpolation for any single-agent normal modal logic can be generalized to its multi-agent case. Pattinson [38] showed that all rank-1 modal logics over a finite number of agents have uniform interpolation. A modal logic is rank-1 if it can be axiomatized by formulas whose modal nesting depth is uniformly equal to one. Thus rank-1 logics include K and D, but exclude other systems such as T, K45, KD45 and S5. So it is known that K_n, D_n, T_n, and $\mathsf{S5}_n$ have uniform interpolation. D'Agostino and Hollenberg [7] showed that the μ-calculus [27], an extension of K_n with a fixed-point operator, has uniform interpolation; later, D'Agostino and Lenzi [8] showed that for a normal form for the μ-calculus, the so-called disjunctive formulas introduced by Janin and Walukiewicz [24], uniform interpolants can be simply obtained by the operation of literal elimination. Studer [39] showed that KC, which is $\mathsf{K_n}$ with common knowledge, does not have uniform interpolation, and neither does K4C. The reason can be explained as follows: the common knowledge operator is defined via an accessibility relation which is the transitive closure of the union of the accessibility relations for all agents. In $\mathsf{K_n}$ and $\mathsf{K4_n}$, this relation satisfies only transitivity. Hence, similarly to K4, neither KC nor K4C has uniform interpolation. In summary, it remains open whether $\mathsf{K45_n}$ and $\mathsf{KD45_n}$ have uniform interpolation, and whether $\mathsf{D_n}$, $\mathsf{T_n}$, $\mathsf{K45_n}$, $\mathsf{KD45_n}$ and $\mathsf{S5_n}$ with common knowledge have uniform interpolation.

In this paper, we systematically study forgetting in multi-agent modal logics. We adopt the semantic definition of existential bisimulation quantifiers as that of forgetting. We resort to a normal form of propositional modal logics, called canonical formulas by Moss [37]. An arbitrary modal formula is equivalent to a disjunction of a finite set of satisfiable canonical formulas, and there is an algorithm to construct this set for any modal system whose satisfiability is decidable. Due to the distributive law of forgetting over disjunction, forgetting for arbitrary modal formulas can be reduced to forgetting for satisfiable canonical formulas. We give a model-theoretic proof that literal elimination generates the result of forgetting for satisfiable canonical formulas: given a model satisfying the formula obtained from a satisfiable canonical formula via literal elimination, we construct a model satisfying the original formula and p-similar to the given model. Thus with a uniform and constructive proof method, we show that except $\mathsf{K4_n}$ and $\mathsf{S4_n}$, which do not have uniform interpolation, the other main multi-agent modal systems, namely $\mathsf{K_n}$, $\mathsf{D_n}$, $\mathsf{T_n}$, $\mathsf{K45_n}$, $\mathsf{KD45_n}$ and $\mathsf{S5_n}$, are closed under forgetting and hence have uniform interpolation; so we settle the open problem concerning $\mathsf{K45_n}$ and $\mathsf{KD45_n}$. Finally, using the same proof methods, we generalize these results to the above modal logics with common knowledge of propositional formulas. To this end, we propose canonical formulas with propositional common knowledge, in short pc-canonical formulas, and show that every formula with propositional common knowledge can be equivalently transformed into a disjunction of satisfiable pc-canonical formulas.

In summary, in this paper, for the systems $\mathsf{K_n}$, $\mathsf{D_n}$, $\mathsf{T_n}$, $\mathsf{K45_n}$, $\mathsf{KD45_n}$ and $\mathsf{S5_n}$ with propositional common knowledge, we provide a uniform method to compute uniform interpolants. However, although our proof is constructive, it has non-elementary complexity since the number and size of canonical formulas is non-elementary. Two important topics are left for future research. The first is more efficient approaches for forgetting and the second is forgetting for more general cases of common knowledge.

A preliminary version of this paper was published in IJCAI-2016 [12]. In this version, we have added the results of forgetting for propositional common knowledge, and full proofs of all propositions, lemmas, and theorems.

The paper is organized as follows. Preliminaries are introduced in Section 2. In Section 3, we define forgetting in multi-agent modal logics and analyze its properties. In Section 4, we show that $\mathsf{K_n}$, $\mathsf{D_n}$, $\mathsf{T_n}$, $\mathsf{K45_n}$, $\mathsf{KD45_n}$ and $\mathsf{S5_n}$ are closed under forgetting. In Section 5, we extend the above results to include propositional common knowledge. Related work is discussed in Section 6. Finally, we conclude the paper. For coherence of presentation, the main theorems in the paper are accompanied by proof sketches only. Details of proofs and yet other proofs can be found in Appendix A.

2. Preliminaries

In this section, we introduce the background material, *i.e.*, the syntax and semantics of multi-agent modal logics, and the canonical formulas for modal logics as defined by Moss [37].

2.1. Multi-agent modal logics

We fix a set $\mathcal{A}$ of n agents and a countable set $\mathcal{P}$ of atoms. A (propositional) literal is an atom p (positive literal) or its negation $\neg p$ (negative literal). Given a finite subset P of $\mathcal{P}$, a *minterm* of P is a conjunction of literals that uses only atoms of P and where each atom in P appears exactly once.

Definition 1. The language $\mathcal{L}^{\mathbf{K}}_{\mathbf{C}}$ is generated by the BNF:

$$\phi ::= p \mid \neg\phi \mid \phi \wedge \phi \mid \phi \vee \phi \mid \mathbf{K}_i\phi \mid \mathbf{C}\phi,$$

where $p \in \mathcal{P}$ and $i \in \mathcal{A}$. We use $\mathcal{L}^{pl}$ for the propositional language, $\mathcal{L}^{\mathbf{K}}_{n}$ for the language without the $\mathbf{C}$ modality, and $\mathcal{L}^{\mathbf{K}}_{\mathbf{PC}}$ for the language with propositional common knowledge where any ϕ appearing in $\mathbf{C}\phi$ must be propositional.

Intuitively, $\mathbf{K}_i\phi$ means that agent i knows ϕ holds, and $\mathbf{C}\phi$ means ϕ is common knowledge among all agents, *i.e.*, everybody knows ϕ, everybody knows everybody knows ϕ, everybody knows everybody knows everybody knows ϕ, and so on. We let $\top$ and $\bot$ represent *true* and *false* respectively. We use i, j and k to range over agents, p and q to range over atoms, ϕ and ψ to range over formulas, and Φ and Ψ to range over finite sets of formulas. We let $\mathcal{P}(\phi)$ denote the set of atoms which appear in ϕ.

The *(modal) depth* of a formula ϕ in $\mathcal{L}^{\mathbf{K}}_{\mathbf{C}}$, written $dep(\phi)$, is the depth of nesting of modal operators in ϕ. We let $\hat{\mathbf{K}}_i\phi$ stand for $\neg\mathbf{K}_i\neg\phi$ and $\hat{\mathbf{C}}\phi$ stand for $\neg\mathbf{C}\neg\phi$. We let $\bigvee\Phi$ (resp. $\bigwedge\Phi$) denote the disjunction (resp. conjunction) of members of Φ; and we use $\bigwedge\hat{\mathbf{K}}_i\Phi$ (resp. $\bigwedge\hat{\mathbf{C}}\Phi$) to abbreviate $\bigwedge_{\phi\in\Phi}\hat{\mathbf{K}}_i\phi$ (resp. $\bigwedge_{\phi\in\Phi}\hat{\mathbf{C}}\Phi$).

Table 1
The main modal systems.

L	K	D	T	K4	S4	K45	KD45	S5
Serial		✓	✓		✓		✓	✓
Reflexive			✓		✓			✓
Symmetric								✓
Transitive				✓	✓	✓	✓	✓
Euclidean						✓	✓	✓

Definition 2. A frame is a pair (S, R), where

- S is a non-empty set of possible worlds;
- R is a function that maps each agent i to a binary relation R_i on S, called the *accessibility relation* for i.

We let $R_{\mathcal{A}}$ denote the transitive closure of the union of R_i for $i \in \mathcal{A}$. For $s \in S$, we write $R_i(s)$ for $\{t \in S \mid sR_it\}$, and call it the set of the i-children of s; similarly, we write $R_{\mathcal{A}}(s)$ for $\{t \in S \mid sR_{\mathcal{A}}t\}$, and call it the set of the descendants of s.

Different *modal systems* result from different sets of conditions on the accessibility relations. We say R_i is *serial* if for any $s \in S$, there is $s' \in S$ s.t. sR_is'; we say R_i is *reflexive* if for any $s \in S$, we have sR_is; we say R_i is *symmetric* if whenever sR_is', we have $s'R_is$; we say R_i is *transitive* if whenever sR_is_1 and $s_1R_is_2$, we have sR_is_2; we say R_i is *Euclidean* if whenever sR_is_1 and sR_is_2, we have $s_1R_is_2$.

In Table 1, we list the modal systems of K, T, K4, S4, KD45, and S5 examined by Halpern and Moses [21], and also include the D and K45 systems. S5 and KD45 are well-accepted as the logics for knowledge and belief, respectively. For each symbol L listed here, we use L for the single agent case, $\mathsf{L_n}$ for the case where there are n agents but no common knowledge, LC for the common knowledge case, and $\mathsf{L_nPC}$ for the propositional common knowledge case.

An interpretation of propositional logic, called a valuation, is a mapping from $\mathcal{P}$ to the values of true and false. We denote a valuation by a subset P of $\mathcal{P}$, meaning that an atom is mapped to true iff it is in P.

Definition 3. A Kripke model is a triple $M = \langle S, R, V\rangle$, where (S, R) is a frame, and V is a valuation map, which maps each $s \in S$ to a valuation. A pointed Kripke model is a pair (M, s), where M is a Kripke model and s is a world of M, called the *actual world*.

For simplicity, we often omit the word "pointed".

Definition 4. Let (M, s) be a Kripke model where $M = \langle S, R, V\rangle$. We interpret formulas in $\mathcal{L}^{\mathbf{K}}_{\mathbf{C}}$ by induction:

- $M, s \models p$ iff $p \in V(s)$;
- $M, s \models \neg\phi$ iff $M, s \not\models \phi$;
- $M, s \models \phi \wedge \psi$ iff $M, s \models \phi$ and $M, s \models \psi$;
- $M, s \models \phi \vee \psi$ iff $M, s \models \phi$ or $M, s \models \psi$;
- $M, s \models \mathbf{K}_i\phi$ iff for all $t \in R_i(s)$, $M, t \models \phi$;
- $M, s \models \mathbf{C}\phi$ iff for all $t \in R_{\mathcal{A}}(s)$, $M, t \models \phi$.

We use L to range over modal systems. Consider the context of a modal system L. We say ϕ is satisfiable, if there exists a Kripke model (M, s) s.t. $M, s \models \phi$. We say ϕ is valid, if for every Kripke model (M, s), we have $M, s \models \phi$. We say ϕ entails ψ, written $\phi \models \psi$, if for any Kripke model (M, s), $M, s \models \phi$ implies $M, s \models \psi$. We say ϕ and ψ are equivalent, written $\phi \equiv \psi$, if $\phi \models \psi$ and $\psi \models \phi$.

2.2. Canonical formulas

Moss [37] defined a particular kind of modal formulas, called *canonical formulas*. A canonical formula captures a Kripke model up to a given depth and can be considered as the analogy in modal logics of a term in propositional logic. Moss [37] gave constructive proofs of completeness of most standard modal logics via canonical formulas. In this paper, we will resort in our proofs to canonical formulas. In this subsection, we introduce the definition of canonical formulas of $\mathcal{L}^{\mathbf{K}}_n$, and present relevant results from Moss [37] that are needed in the proofs of our paper.

The canonical formulas can be conveniently defined using a single modality – the cover modality, which was introduced by Janin and Walukiewicz [24] for the μ-calculus. So we first introduce the cover modality for multi-agent modal logics. Intuitively, $\nabla_i\Phi$ means that Φ covers all formulas considered possible by agent i.

Definition 5. Let $i \in \mathcal{A}$, and Φ be a finite set of formulas in $\mathcal{L}^{\mathbf{K}}_{\mathbf{C}}$. The cover modality is defined as follows:

$$\nabla_i\Phi \doteq \mathbf{K}_i(\bigvee \Phi) \wedge (\bigwedge \hat{\mathbf{K}}_i\Phi).$$

Note that $\nabla_i \emptyset \equiv \mathbf{K}_i \bot$.

It is easy to prove the following:

Proposition 1. *Let (M, s) be a Kripke model where $M = \langle S, R, V \rangle$ and $s \in S$. Then, $M, s \models \nabla_i \Phi$ iff the following conditions hold:*

Forth *For all $t \in R_i(s)$, there is $\phi \in \Phi$ s.t. $M, t \models \phi$;*
Back *For all $\phi \in \Phi$, there is $t \in R_i(s)$ s.t. $M, t \models \phi$.*

Thus each i-child of s satisfies some member of Φ, and each member of Φ is satisfied by some i-child of s.

The following proposition can be used to convert an arbitrary formula in $\mathcal{L}_n^{\mathbf{K}}$ to a formula using only the cover modalities.

Proposition 2. *Suppose that ϕ is $\bot$ when Ψ is empty. Then*

$$\mathbf{K}_i \phi \wedge (\bigwedge \hat{\mathbf{K}}_i \Psi) \equiv \nabla_i cov(\phi, \Psi), \text{ where}$$

$$cov(\phi, \Psi) \doteq \begin{cases} \emptyset, & \text{if } \phi = \bot; \\ \{\phi\} \cup \{\phi \wedge \psi \mid \psi \in \Psi\}, & \text{otherwise.} \end{cases}$$

Intuitively, $\mathrm{cov}(\phi, \Psi)$ covers all the possibilities for agent i.

Example 1. Let $\phi = \mathbf{K}_i p \wedge \hat{\mathbf{K}}_i q \wedge \hat{\mathbf{K}}_i \neg q$, meaning that agent i knows p but is ignorant about q. By Proposition 2, $\phi \equiv \nabla_i \{p, p \wedge q, p \wedge \neg q\}$.

Now we use the cover modality to define canonical formulas. Intuitively, a canonical formula captures a Kripke model (M, s) up to a given depth and a finite set P of atoms: a depth 0 canonical formula capturing (M, s) is simply a minterm of atoms from P, describing the actual world s; a depth $(k+1)$ canonical formula capturing (M, s) is a formula of the form $\delta_0 \wedge \bigwedge_{i \in \mathcal{A}} \nabla_i \Phi_i$, where δ_0 is a depth 0 canonical formula capturing (M, s), and for each agent i, Φ_i is the set of depth k canonical formulas capturing (M, t) for some i-child t of s.

Let S_1 and S_2 be two sets. We use $S_1 \setminus S_2$ to denote the difference of S_1 and S_2. Let S be a finite set. We use $|S|$ to denote the cardinality of S.

Definition 6 *(Canonical formulas).* Let $P \subseteq \mathcal{P}$ be finite. We inductively define the set E_k^P as follows:

- $E_0^P = \{\bigwedge_{p \in S} p \wedge \bigwedge_{p \in P \setminus S} \neg p \mid S \subseteq P\}$, *i.e.*, E_0^P is the set of minterms of P;
- $E_{k+1}^P = \{\delta_0 \wedge \bigwedge_{i \in \mathcal{A}} \nabla_i \Phi_i \mid \delta_0 \in E_0^P \text{ and } \Phi_i \subseteq E_k^P\}$.

We call each member of E_k^P a canonical formula with depth k and alphabet P.

Let $\delta = \delta_0 \wedge \bigwedge_{i \in \mathcal{A}} \nabla_i \Phi_i \in E_{k+1}^P$. We denote δ_0 by $w(\delta)$, and call it the world of δ; we denote Φ_i by $R_i(\delta)$, and call it the set of the i-children of δ. For a set Φ of canonical formulas, we use $w(\Phi)$ to denote the set $\{w(\phi) \mid \phi \in \Phi\}$.

Let $|P| = m$ and $|\mathcal{A}| = n$. We define $F(0, m) = 2^m$ and $F(k+1, m) = 2^{n \cdot F(k,m) + m}$. Clearly, $|E_k^P| = F(k, m)$. Thus the number and size of canonical formulas are non-elementary in the number of atoms [37].

Given a modal system L, there may exist unsatisfiable canonical formulas. For example, $p \wedge \nabla_1 \{\neg p\}$ is unsatisfiable in the system T. We use $E_k^P(\mathsf{L})$ to denote the set of canonical formulas satisfiable in L.

A satisfiable canonical formula δ of E_k^P is a consistent complete theory w.r.t. the set of atoms P and the depth k. That is, for any formula ϕ which uses only atoms of P and whose depth is at most k, δ entails either ϕ or its negation.

Proposition 3 *(Moss [37]). Consider the context of a modal system L. Let $\delta \in E_k^P(\mathsf{L})$ where $k \in \mathbb{N}$ and $P \subseteq \mathcal{P}$ is finite. Let $\phi \in \mathcal{L}_n^{\mathbf{K}}$ s.t. $dep(\phi) \le k$ and $\mathcal{P}(\phi) \subseteq P$. Then either $\delta \models \phi$ or $\delta \models \neg\phi$.*

Moreover, the following proposition gives us an algorithm to check if $\delta \models \phi$.

Proposition 4. *Consider the context of a modal system L. Let $\delta \in E_k^P(\mathsf{L})$ where $k \in \mathbb{N}$ and $P \subseteq \mathcal{P}$ is finite. Let $\phi \in \mathcal{L}_n^{\mathbf{K}}$ s.t. $dep(\phi) \le k$ and $\mathcal{P}(\phi) \subseteq P$. Then we can check if $\delta \models \phi$ recursively as follows:*

- *$\delta \models p$ iff p appears positively in $w(\delta)$;*
- *$\delta \models \neg\phi$ iff $\delta \not\models \phi$;*
- *$\delta \models \phi \wedge \psi$ iff $\delta \models \phi$ and $\delta \models \psi$;*
- *$\delta \models \phi \vee \psi$ iff $\delta \models \phi$ or $\delta \models \psi$;*
- *$\delta \models \mathbf{K}_i \phi$ iff for all $\delta' \in R_i(\delta)$, $\delta' \models \phi$.*

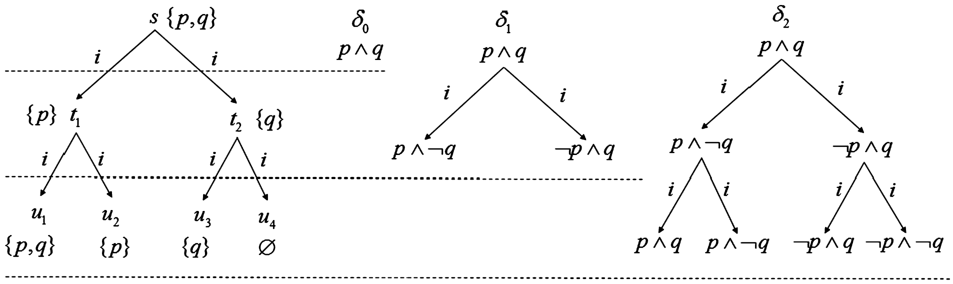

Fig. 1. Illustration of Proposition 5.

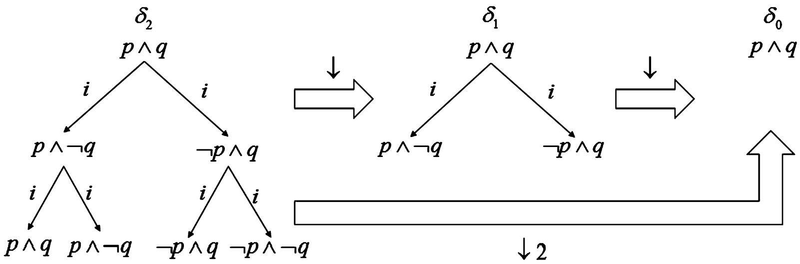

Fig. 2. The projection operation on canonical formulas.

The following proposition says that every Kripke model (M, s) satisfies a unique canonical formula of a given depth k, which we call the depth k canonical formula of (M, s).

Proposition 5 *(Moss [37]). Let (M, s) be a Kripke model and $k \in \mathbb{N}$. Let $P \subseteq \mathcal{P}$ be finite. Then, there exists a unique $\delta \in E_k^P$ s.t. $M, s \models \delta$.*

We illustrate this proposition with an example.

Example 2. Fig. 1 shows a Kripke model (M, s) on the left and the relevant depth k canonical formulas δ_k's $(k = 0, 1, 2)$ on the right. Here $\delta_0 = p \wedge q$, $\delta_1 = p \wedge q \wedge \nabla_i\{p \wedge \neg q, \neg p \wedge q\}$, and $\delta_2 = p \wedge q \wedge \nabla_i\{p \wedge \neg q \wedge \nabla_i\{p \wedge q, p \wedge \neg q\}, \neg p \wedge q \wedge \nabla_i\{\neg p \wedge q, \neg p \wedge \neg q\}\}$.

The following proposition says that every modal formula is equivalent to the disjunction of a unique set of satisfiable canonical formulas whose depth is not less than that of the original formula.

Proposition 6 *(Moss [37]). Consider the context of a modal system L. Let $\phi \in \mathcal{L}_n^{\mathbf{K}}$, $k \geq dep(\phi)$ and $P = \mathcal{P}(\phi)$. Then there exists a unique set $\Phi \subseteq E_k^P(\mathsf{L})$ s.t. $\phi \equiv \bigvee \Phi$.*

In fact, for the modal systems listed in Table 1, there is an algorithm to construct Φ in the above proposition, although the algorithm is of non-elementary complexity. Firstly, we construct E_k^P. Then we remove from it those formulas unsatisfiable in L: for all the modal systems in Table 1, satisfiability is decidable [21]. Finally, by using Proposition 4, we remove from $E_k^P(\mathsf{L})$ those formulas which entail $\neg\phi$.

Finally, we introduce the projection operations on canonical formulas that are needed in the proofs of our paper.

As shown in Fig. 1, a canonical formula can be graphically represented as a tree. The operation $\delta^{\downarrow}$ prunes the leaves of this tree, while $\delta^{\downarrow l}$ prunes the bottom l levels of the tree. We call $\delta^{\downarrow}$ the 1st-cut of δ, and $\delta^{\downarrow l}$ the lth-cut of δ. Fig. 2 depicts that $\delta_2^{\downarrow} = \delta_1$ and $\delta_2^{\downarrow 2} = \delta_0$.

Definition 7. Let $P \subseteq \mathcal{P}$ be finite. Let $k \in \mathbb{N}$ and $\delta \in E_k^P$. Then, $\delta^{\downarrow}$ is inductively defined as follows:

$$\delta^{\downarrow} = \begin{cases} \delta, & \text{if } k = 0; \\ w(\delta), & \text{if } k = 1; \\ w(\delta) \wedge \bigwedge_{i \in \mathcal{A}} \nabla_i (R_i(\delta))^{\downarrow}, & \text{otherwise.} \end{cases}$$

Let Φ be a set of canonical formulas. We use $\Phi^{\downarrow}$ to denote the set $\{\phi^{\downarrow} \mid \phi \in \Phi\}$.

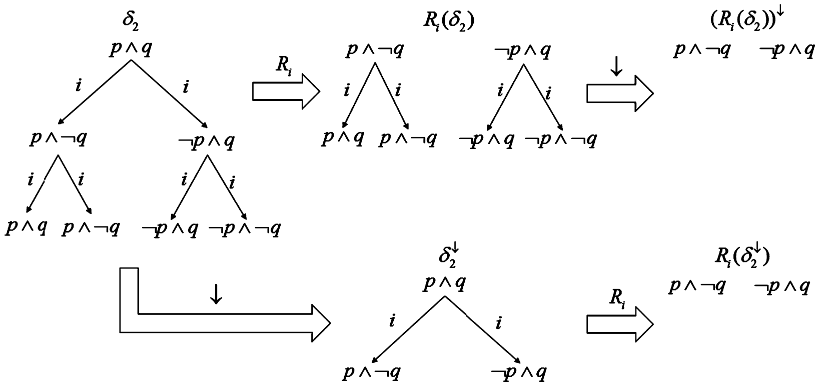

Fig. 3. An example of Proposition 7.

Definition 8. Let $P \subseteq \mathcal{P}$ be finite. Let $k, l \in \mathbb{N}$ s.t. $k \geq l$. Let $\delta \in E_k^P$. Then, $\delta^{\downarrow l}$ is inductively defined as follows:

$$\delta^{\downarrow l} = \begin{cases} \delta, & \text{if } l = 0; \\ (\delta^{\downarrow l-1})^{\downarrow}, & \text{otherwise.} \end{cases}$$

Similarly to $\Phi^{\downarrow}$, $\Phi^{\downarrow l}$ is the set $\{\phi^{\downarrow l} \mid \phi \in \Phi\}$. By induction, it is easy to prove that $\delta \models \delta^{\downarrow l}$ for all $l \leq dep(\delta)$.

The following proposition says that the i-children of the lth-cut of a canonical formula are equal to the lth-cut of its i-children, *i.e.*, the two operations are commutative.

Proposition 7. *Let δ be a canonical formula and $l < dep(\delta)$. Then for all $i \in \mathcal{A}$, we have $R_i(\delta^{\downarrow l}) = (R_i(\delta))^{\downarrow l}$.*

Example 3. Let us continue Example 2. As illustrated in Fig. 3, we get the following:

- $R_i(\delta_2) = \{p \wedge \neg q \wedge \nabla_i\{p \wedge q, p \wedge \neg q\}, \neg p \wedge q \wedge \nabla_i\{\neg p \wedge q, \neg p \wedge \neg q\}\}$;
- $(R_i(\delta_2))^{\downarrow} = \{p \wedge \neg q, \neg p \wedge q\}$;
- $\delta_2^{\downarrow} = p \wedge q \wedge \nabla_i\{p \wedge \neg q, \neg p \wedge q\}$;
- $R_i(\delta_2^{\downarrow}) = \{p \wedge \neg q, \neg p \wedge q\}$.

Hence, $(R_i(\delta_2))^{\downarrow 1} = R_i(\delta_2^{\downarrow 1})$.

3. Definition of forgetting

In this section, we define forgetting in multi-agent modal logics and analyze its properties.

We first review Lin and Reiter's definition of forgetting in first-order logic.

Definition 9. Let P be a predicate, and let M_1 and M_2 be two structures. We say M_1 and M_2 are P-identical, written $M_1 \sim_P M_2$, if M_1 and M_2 agree on everything except possibly on the interpretation of P.

Definition 10. Let T be a theory, and P a predicate. A theory T' is a result of forgetting P in T, denoted by $\text{forget}(T, P) \equiv T'$, if for any structure M, $M \models T'$ iff there is a model M' of T such that $M \sim_P M'$.

We now apply Lin and Reiter's definition to modal logics. The set $\mathcal{A}$ of agents and set $\mathcal{P}$ of atoms induce a first-order language consisting of the following: A unary predicate $p(s)$ for each $p \in \mathcal{P}$; A binary predicate $R_i(s, s')$ for each $i \in \mathcal{A}$; A constant S_0 indicating the actual world. A pointed Kripke model (M, s) where $M = \langle S, R, V \rangle$ induces a first-order structure M_s as follows: The domain is S; the interpretation of $p(s)$ is the set of states where p holds; the interpretation of $R_i(s, s')$ is R_i, and the interpretation of S_0 is s. We write $M, s \sim_p M', s'$ if $M_s \sim_p M'_{s'}$. Thus $M, s \sim_p M', s'$ means that the two pointed Kripke models are p-identical: they have the same frame, the same actual world, and the same interpretation of every atom except p.

Applying Lin and Reiter's definition of forgetting to modal logics, we get a notion of forgetting which is too strong. To illustrate, let ϕ be $\hat{\mathbf{K}}_i p \wedge \hat{\mathbf{K}}_i \neg p$, meaning that agent i is ignorant about p. Then the result of forgetting p in ϕ states that the actual world have at least two different i-children, which is not expressible in $\mathcal{L}_n^{\mathbf{K}}$. However, a uniform interpolant of ϕ w.r.t. $\emptyset$ is $\hat{\mathbf{K}}_i \top$.

To give an appropriate definition of forgetting in multi-agent modal logics, we weaken the condition that two Kripke models are identical. As mentioned in the introduction, bisimulation quantifiers were introduced by Visser [41] and Ghilardi and Zawadowski [18] to semantically characterize uniform interpolants for modal logics. A model M satisfies a formula $\exists p\phi$ where p is an atom iff there is a model M' satisfying ϕ such that M and M' are p-bisimilar. We weaken "identical" to "bisimilar", and adopt the semantics of existential bisimulation quantifiers as that of forgetting.

Definition 11 *(p-bisimulation).* Let (M, s) and (M', s') be two Kripke models where $M = \langle S, R, V\rangle$ and $M' = \langle S', R', V'\rangle$. A p-bisimulation between (M, s) and (M', s') is a relation $\rho \subseteq S \times S'$ s.t. $s\rho s'$, and whenever $t\rho t'$, we get:

atoms $V(t) \sim_p V'(t')$;
forth For all i, if tR_iu, then there is u' s.t. $t'R'_iu'$ and $u\rho u'$;
back For all i, if $t'R'_iu'$, then there is u s.t. tR_iu and $u\rho u'$.

We say that (M, s) and (M', s') are p-bisimilar, written $(M, s)\underline{\leftrightarrow}_p(M', s')$, if there is a p-bisimulation between them.

It can be easily proved that $\underline{\leftrightarrow}_p$ is an equivalence relation [14]. A nice property of p-bisimilar Kripke models is that they agree on all modal formulas wherein p does not appear.

Proposition 8. *Let $\phi \in \mathcal{L}^{\mathbf{K}}_{\mathbf{C}}$ wherein p does not appear. Then, $(M, s)\underline{\leftrightarrow}_p(M', s')$ implies that $M, s \models \phi$ iff $M', s' \models \phi$.*

We now define forgetting in multi-agent modal logics. Note two differences from Lin and Reiter's definition: First, we are concerned with forgetting in finite theories, *i.e.*, formulas, rather than arbitrary theories. Second, we require that when forgetting p in ϕ, p should not appear in the resulting formula.

Definition 12 *(Forgetting).* Consider the context of a modal system L. Let $\phi \in \mathcal{L}^{\mathbf{K}}_{\mathbf{C}}$ and p an atom. A formula ψ s.t. $\mathcal{P}(\psi) \subseteq \mathcal{P}(\phi) \setminus \{p\}$ is a result of forgetting p in ϕ, written $\text{kforget}(\phi, p) \equiv \psi$, if the following conditions hold:

forth For any Kripke models (M, s) and (M', s'), if $M, s \models \phi$ and $(M, s)\underline{\leftrightarrow}_p(M', s')$, then $M', s' \models \psi$;
back For any model (M', s') of ψ, there is a model (M, s) of ϕ s.t. $(M, s)\underline{\leftrightarrow}_p(M', s')$.

By the above definition, if ψ is a result of forgetting p in ϕ, then $M', s' \models \psi$ iff there is a model (M, s) of ϕ s.t. $(M, s)\underline{\leftrightarrow}_p(M', s')$. Thus, if both ψ_1 and ψ_2 are results of forgetting p in ϕ, then they are logically equivalent.

Consider again $\phi = \hat{\mathbf{K}}_i p \wedge \hat{\mathbf{K}}_i \neg p$. We show that $\text{kforget}(\phi, p) \equiv \hat{\mathbf{K}}_i \top$. This is because $M', s' \models \hat{\mathbf{K}}_i \top$ iff there is (M, s) s.t. $M, s \models \hat{\mathbf{K}}_i p \wedge \hat{\mathbf{K}}_i \neg p$ and $(M, s)\underline{\leftrightarrow}_p(M', s')$.

We now analyze basic properties of forgetting.

Proposition 9. *Consider the context of a modal system L. If kforget$(\phi, p) \equiv \psi$, then the following hold:*

1. $\phi \models \psi$;
2. *for any formula η wherein p does not appear, $\phi \models \eta$ iff $\psi \models \eta$.*

Proposition 9 says that forgetting a proposition p from a formula ϕ yields a formula ψ which is weaker than ϕ; and for any query wherein p does not appear, ϕ and ψ are equivalent.

Proposition 10. *Consider the context of a modal system L. The following hold:*

1. *If $\phi \in \mathcal{L}^{pl}$ and forget$(\phi, p) \equiv \psi$, then kforget$(\phi, p) \equiv \psi$;*
2. *kforget$(\phi_1 \vee \phi_2, p) \equiv$ kforget$(\phi_1, p) \vee$ kforget(ϕ_2, p).*

Proposition 10 says that if the original formula is propositional, then the result of forgetting in modal logic is the same as that in propositional logic; and forgetting is distributive over disjunction.

We now relate forgetting to uniform interpolation.

Definition 13. We say that a modal system L is closed under forgetting if for any formula ϕ and any atom $p \in \mathcal{P}(\phi)$, there exists an L formula ψ s.t. $\text{kforget}(\phi, p) \equiv \psi$.

Definition 14. We say a modal system L has uniform interpolation if for every formula ϕ and every $P \subseteq \mathcal{P}(\phi)$, there is a formula ψ such that $\mathcal{P}(\psi) \subseteq P$ and such that for any formula η with $\mathcal{P}(\eta) \subseteq P$, we have $\phi \models \eta$ iff $\psi \models \eta$. We say ψ is a uniform interpolant of ϕ w.r.t. P.

By Proposition 9, we have

Proposition 11. *If a modal system* L *is closed under forgetting, then* L *has uniform interpolation.*

Zhang and Zhou [48] proposed four postulates for forgetting in S5 as follows.

Definition 15. Let $\phi \in \mathcal{L}_n^{\mathbf{K}}$ and p an atom. We say that ϕ is irrelevant to p, written $\mathrm{IR}(\phi, p)$, if there exists a formula ψ s.t. $\phi \equiv \psi$ and p does not appear in ψ.

Definition 16 *(Forgetting postulates [48]).* Let $\phi, \psi \in \mathcal{L}_n^{\mathbf{K}}$ and p an atom. We say ψ is a result of forgetting p from ϕ, if the following postulates are satisfied:

(W) Weakening: $\phi \models \psi$;
(PP) Positive Persistence: if $\mathrm{IR}(\eta, p)$ and $\phi \models \eta$, then $\psi \models \eta$;
(NP) Negative Persistence: if $\mathrm{IR}(\eta, p)$ and $\phi \not\models \eta$, then $\psi \not\models \eta$;
(IR) Irrelevance: $\mathrm{IR}(\psi, p)$.

It is easy to see that the four postulates coincide with the definition of uniform interpolants w.r.t. atoms other than p modulo irrelevance.

By Proposition 11, closure under forgetting (elimination of bisimulation quantifiers) implies uniform interpolation. D'Agostino and Lenzi [9] showed that the converse is not true in general, but provided a proof in the case of finite transitive frames.

As mentioned in the introduction, none of K4, S4, KC and K4C has uniform interpolation. Since K4-models satisfy transitivity, $\mathsf{K4_n}$, $\mathsf{K4_nPC}$ and K4C are equally expressive in the single-agent case. For $\mathsf{S4_n}$, $\mathsf{S4_nPC}$ and S4C, the above fact also holds. Hence, none of $\mathsf{K4_nPC}$, $\mathsf{S4_nPC}$ and S4C has uniform interpolation.

Corollary 1. *None of* $\mathsf{K4_n}$, $\mathsf{S4_n}$, $\mathsf{K4_nPC}$, $\mathsf{S4_nPC}$, KC, K4C *and* S4C *is closed under forgetting.*

Finally, we propose a syntactical method of forgetting called literal elimination. As mentioned in the introduction, D'Agostino and Lenzi [8] showed that it computes uniform interpolants for disjunctive formulas for the μ-calculus. In the following sections, we will show that it computes the result of forgetting for satisfiable canonical formulas for the logics of $\mathsf{K_n}$, $\mathsf{D_n}$, $\mathsf{T_n}$, $\mathsf{K45_n}$, $\mathsf{KD45_n}$ and $\mathsf{S5_n}$.

Definition 17 *(Literal elimination).* Let $\phi \in \mathcal{L}_{\mathbf{C}}^{\mathbf{K}}$ and p an atom. We let ϕ^p denote the formula obtained from ϕ by substituting all occurrences of $\neg p$ with $\top$ and subsequently substituting all occurrences of p with $\top$.

Similarly to $\Phi^{\downarrow}$, Φ^p is the set $\{\phi^p \mid \phi \in \Phi\}$.

Merely requiring to replace $\neg p$ and p by $\top$ would be ambiguous about whether we should replace $\neg p$ by $\top$ or by $\bot$. The latter would happen if we were to replace the p in $\neg p$ by $\top$. That would be an undesirable outcome. The "subsequently" in Definition 17 is not ambiguous and avoids that outcome.

Example 4. Let $\delta_1 = p \wedge q \wedge \nabla_i\{\neg p \wedge \neg q\}$. Then, $\delta_1^p = \top \wedge q \wedge \nabla_i\{\top \wedge \neg q\} \equiv q \wedge \nabla_i\{\neg q\}$.

It is easy to get a property that literal elimination weakens a canonical formula.

Proposition 12. *Let δ be a canonical formula and p an atom. Then, $\delta \models \delta^p$.*

Finally, we comment that for K4 and S4, literal elimination does not necessarily compute the result of forgetting for satisfiable canonical formulas. Recall that in the introduction, we mentioned that in S4, $\exists p \exists q \phi$ where

$$\phi = p \wedge \mathbf{K}(p \to \hat{\mathbf{K}}q) \wedge \mathbf{K}(q \to \hat{\mathbf{K}}p) \wedge \mathbf{K}(q \to r) \wedge \mathbf{K}(p \to \neg r)$$

is equivalent to the following infinite set of formulas:

$$\{\neg r, \hat{\mathbf{K}}r, \hat{\mathbf{K}}(r \wedge \hat{\mathbf{K}}\neg r), \hat{\mathbf{K}}(r \wedge \hat{\mathbf{K}}(\neg r \wedge \hat{\mathbf{K}}r)), \cdots\},$$

which is not equivalent to any quantifier-free formula. Assume that for S4, literal elimination does compute the result of forgetting for satisfiable canonical formulas. We first put ϕ into the disjunction of satisfiable canonical formulas. Then we eliminate the literals of p and q. Since forgetting distributes over disjunction, we get a formula which is the result of forgetting p and q in ϕ. Since we adopt the semantic definition of existential bisimulation quantifiers as that of forgetting, this quantifier-free formula is equivalent to $\exists p \exists q \phi$. This is a contradiction.

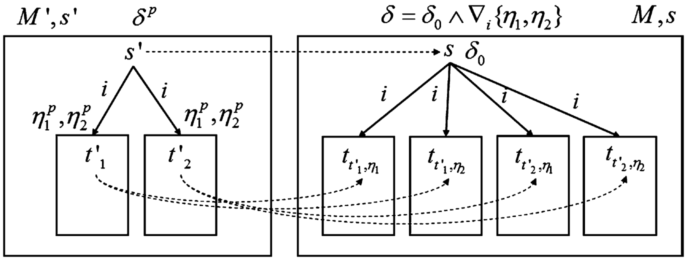

Fig. 4. Illustration for the proof of Theorem 1.

4. Forgetting via literal elimination

In this section, we show that forgetting from satisfiable canonical formulas can be computed via literal elimination in the following multi-agent modal logics: $\mathsf{K_n}$, $\mathsf{D_n}$, $\mathsf{T_n}$, $\mathsf{K45_n}$, $\mathsf{KD45_n}$ and $\mathsf{S5_n}$. As an easy corollary, we have that the above logics are closed under forgetting and that they have uniform interpolation.

4.1. Forgetting in $\mathsf{K_n}$ *and* $\mathsf{D_n}$

In this subsection, we consider $\mathsf{K_n}$ and $\mathsf{D_n}$ cases. The result for $\mathsf{K_n}$ was first proved by Wolter [45]. Here we present an easy inductive proof, which serves as the basis for the proofs of all the forgetting results in this paper.

Theorem 1 *(The basic theorem). Let* L *be* $\mathsf{K_n}$ *or* $\mathsf{D_n}$*, and* δ *an* L*-satisfiable canonical formula. Then* $kforget_{\mathsf{L}}(\delta, p) \equiv \delta^p$.

Proof. We first consider $\mathsf{K_n}$. We prove by induction on $dep(\delta)$.

Base case: We show that the forth and back conditions of the definition of forgetting in modal logics (Definition 12) hold.

Forth: Let (M, s) and (M', s') be two models s.t. $M, s \models \delta$ and $(M, s)\underline{\leftrightarrow}_p(M', s')$. By Proposition 12, we get that $\delta \models \delta^p$, and hence $M, s \models \delta^p$. Since p does not appear in δ^p, by Proposition 8, $M', s' \models \delta^p$.

Back: Let (M', s') be a model of δ^p. We get the valuation $V'(s')$ of the actual world s'. Let M be a copy of M' except the valuation on s'. If $\delta \models p$, then we let $V(s) = V'(s') \cup \{p\}$; otherwise, we let $V(s) = V'(s') \setminus \{p\}$. It is obvious that $M, s \models \delta$ and $(M, s)\underline{\leftrightarrow}_p(M', s')$.

Induction step: Here, we only prove the back condition. The proof of the forth condition is similar to that in the base case. Let $M', s' \models \delta^p$. We construct M and define a relation ρ between the worlds of M and M' as follows. The only tricky part of the construction is that it is possible that there exist $i \in \mathcal{A}$, $t' \in R_i(s')$, $\eta_1, \eta_2 \in R_i(\delta)$ s.t. (M', t') satisfies both η_1^p and η_2^p. Fig. 4 illustrates the construction where $\delta = \delta_0 \wedge \nabla_i\{\eta_1, \eta_2\}$, $R'_i(s') = \{t'_1, t'_2\}$, and both (M', t'_1) and (M', t'_2) satisfy both η_1^p and η_2^p.

1. Create a new world s, let $s\rho s'$, and $V(s) \models w(\delta)$.
2. For all $i \in \mathcal{A}$, $t' \in R'_i(s')$, and $\eta \in R_i(\delta)$, if $M', t' \models \eta^p$, by the induction hypothesis, there exist $(M_{t',\eta}, t_{t',\eta})$ and $\rho_{t',\eta}$ s.t. $M_{t',\eta}, t_{t',\eta} \models \eta$ and $\rho_{t',\eta} : (M_{t',\eta}, t_{t',\eta})\underline{\leftrightarrow}_p(M', t')$. Add a new copy of $M_{t',\eta}$ into M, let $sR_i t_{t',\eta}$, and expand ρ with $\rho_{t',\eta}$.

It is easy to verify that $\rho : (M, s)\underline{\leftrightarrow}_p(M', s')$ and $M, s \models \delta$.

In the case of $\mathsf{D_n}$, it is obvious from the above construction that if M' satisfies seriality, so does M. $\square$

From the above proof, we can observe that the proof of the forth condition of Definition 12 is straightforward by making use of $(M, s)\underline{\leftrightarrow}_p(M', s')$ and Propositions 8 and 12. In addition, the back condition of the base case is easily proved by letting M be a copy of M' and modifying the valuation on s so that $M, s \models \delta$. In the following, when we prove $\mathrm{kforget}(\delta, p) \equiv \delta^p$ in other modal systems, we only present the proof of the back condition of the induction step.

By Proposition 6, every modal formula is equivalent to a disjunction of satisfiable canonical formulas. By Proposition 10, forgetting is distributive over forgetting. Thus by Theorem 1, we get

Corollary 2. $\mathsf{K_n}$ *and* $\mathsf{D_n}$ *are closed under forgetting.*

Example 5. Let ϕ be $\hat{\mathbf{K}}_i p \wedge \hat{\mathbf{K}}_i \neg p$. By Proposition 2, we can convert ϕ into a disjunction of canonical formulas as follows: $\phi \equiv \mathbf{K}_i \top \wedge \hat{\mathbf{K}}_i p \wedge \hat{\mathbf{K}}_i \neg p \equiv \nabla_i\{\top, p, \neg p\} \equiv \delta_1 \vee \delta_2$, where $\delta_1 = p \wedge \nabla_i\{\top, p, \neg p\}$ and $\delta_2 = \neg p \wedge \nabla_i\{\top, p, \neg p\}$. Then $\delta_1^p = \delta_2^p = \top \wedge \nabla_i\{\top\}$. The disjunction of them is equivalent to $\hat{\mathbf{K}}_i \top$.

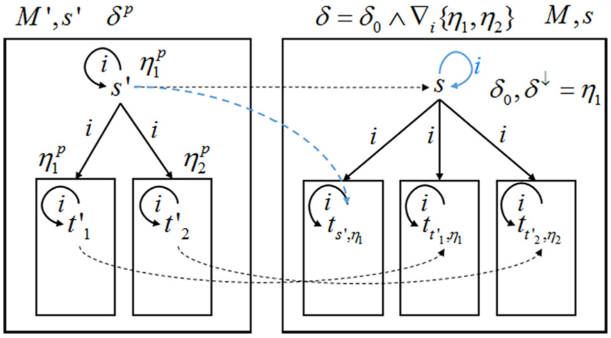

Fig. 5. Illustration for the proof of Theorem 2.

4.2. *Forgetting in* $\mathsf{T_n}$

In this subsection, we analyze properties of $\mathsf{T_n}$ satisfiable canonical formulas, and show that forgetting via literal elimination applies to them.

We begin with a simple example which shows that Theorem 1 does not hold for unsatisfiable canonical formulas.

Example 6. Let $\delta = \neg p \wedge \nabla_i\{p\}$. Clearly, δ is a canonical formula, and it is equivalent to $\bot$ in $\mathsf{T_n}$. However, $\delta^p = \top \wedge \nabla_i\{\top\}$, which is equivalent to $\top$.

The reason that forgetting via literal elimination does not work on δ is that δ is unsatisfiable in $\mathsf{T_n}$. This example illustrates that the literal elimination method is suitable only for satisfiable formulas.

The following proposition says that any $\mathsf{T_n}$ satisfiable canonical formula δ has the reflexive property: for any agent i and any $1 \le l \le dep(\delta)$, the lth-cut of δ is an i-child of its $(l-1)$th-cut.

Proposition 13. *Let δ be a $\mathsf{T_n}$ satisfiable canonical formula where $dep(\delta) \ge 1$. Let $l \in \mathbb{N}$ s.t. $1 \le l \le dep(\delta)$. Then, for all $i \in \mathcal{A}$, we have $\delta^{\downarrow l} \in R_i(\delta^{\downarrow l-1})$.*

Example 6 Cont'd. We have $\delta^{\downarrow} = \neg p$ and $R_i(\delta) = \{p\}$. Obviously, $\delta^{\downarrow} \notin R_i(\delta)$.

Example 7. Let $\delta = p \wedge q \wedge \nabla_i\{p \wedge q, p \wedge \neg q\}$. It is a $\mathsf{T_n}$ satisfiable canonical formula. Then, $\delta^{\downarrow} = p \wedge q$, and $R_i(\delta) = \{p \wedge q, p \wedge \neg q\}$. Obviously, $\delta^{\downarrow} \in R_i(\delta)$.

Theorem 2 *(The $\mathsf{T_n}$ theorem). Let δ be a $\mathsf{T_n}$ satisfiable canonical formula. Then $kforget_{\mathsf{T_n}}(\delta, p) \equiv \delta^p$.*

Proof. The proof is the same as that of the basic theorem except the following. In the induction step via the construction, we get $(M, s) \underline{\leftrightarrow}_p (M', s')$ and $M, s \models \delta$. Let $k = dep(\delta)$. Then for any $l \le k$, $M, s \models \delta^{\downarrow l}$. Although the model is not reflexive, we can fix it via adding the reflexive edge for the world s, *i.e.*, let sR_is for each agent i. Note that $s' \in R'_i(s')$. Fig. 5 shows the case that $M', s' \models \eta_1^p$, so s has an i-child of t_{s',η_1} s.t. $M, t_{s',\eta_1} \models \eta_1$. It is obvious that (M, s) is a $\mathsf{T_n}$ model. It is easy to see that $(M, s) \underline{\leftrightarrow}_p (M', s')$ still holds. It remains to show that $M, s \models \delta$ still holds. We prove by induction on $k - l$ that for any $l \le k$, $M, s \models \delta^{\downarrow l}$ still holds.

Base case: Obviously, $M, s \models w(\delta)$, which is $\delta^{\downarrow k}$.

Induction step: Suppose that $M, s \models \delta^{\downarrow l}$. To show that $M, s \models \delta^{\downarrow l-1}$ still holds, it suffices to show that for each $i \in \mathcal{A}$, there exists $\eta \in R_i(\delta^{\downarrow l-1})$ s.t. $M, s \models \eta$. By the reflexive property (Proposition 13), $\delta^{\downarrow l} \in R_i(\delta^{\downarrow l-1})$. Hence $\delta^{\downarrow l}$ is the desired η. $\square$

4.3. *Multi-pointed Kripke models*

In the next subsection, we show that forgetting via literal elimination applies to satisfiable canonical formulas of $\mathsf{K45_n}$, $\mathsf{KD45_n}$ and $\mathsf{S5_n}$. The proof for the basic theorem does not immediately carry over to these cases, because the model constructed in the proof may not be transitive or Euclidean. Also, we cannot simply fix the problem by adding edges as we do in the proof of the $\mathsf{T_n}$ theorem. We will overcome the problem via multi-pointed Kripke models which offer flexibility in the construction of required models. In this subsection, we introduce the basic concepts regarding multi-pointed Kripke models.

Definition 18. A multi-pointed Kripke model is a pair (M, T) where M is a Kripke model, and T is a possibly empty set of worlds of M.

Throughout this paper, we use (M, s) to denote a single-pointed model, and (M, T) a multi-pointed model.

Given a single-pointed model (M, s) and $i \in \mathcal{A}$, we can naturally obtain a multi-pointed model (M, T) where $T = R_i(s)$, *i.e.*, the i-children of s.

Similarly to the semantics of the cover modality, we have:

Definition 19. Let Φ be a set of formulas. We say a multi-pointed model (M, T) is Φ-*complete* if the following conditions hold

Forth for every $t \in T$, there exists $\phi \in \Phi$ s.t. $M, t \models \phi$;
Back for every $\phi \in \Phi$, there exists $t \in T$ s.t. $M, t \models \phi$.

Obviously, $M, s \models \nabla_i \Phi$ iff $(M, R_i(s))$ is Φ-complete.

Then, we can extend Proposition 5 to multi-pointed models: we have that a multi-pointed model (M, T) corresponds to a unique set of canonical formulas of a given depth k, which we call the depth k canonical formula set of (M, T).

Proposition 14. *Let (M, T) be a multi-pointed model, and $k \in \mathbb{N}$. Let $P \subseteq \mathcal{P}$ be finite. Then, there exists a unique set $\Phi \subseteq E_k^P$ s.t. (M, T) is Φ-complete.*

We now extend the concept of bisimulation to multi-pointed models.

Definition 20. Let (M, T) and (M', T') be two Kripke models. A p-bisimulation between (M, T) and (M', T') is a relation ρ between the worlds of M and M' s.t.

- for every $t \in T$, there exists $t' \in T'$ s.t. $t\rho t'$;
- for every $t' \in T'$, there exists $t \in T$ s.t. $t\rho t'$;
- whenever $u\rho u'$, the conditions atoms, forth and back in Definition 11 hold.

Similarly to Proposition 8, we have a nice property of p-bisimilar multi-pointed models as follows:

Proposition 15. *Let $\Phi \subseteq \mathcal{L}_{\mathbf{C}}^{\mathbf{K}}$ such that Φ is finite and every $\phi \in \Phi$ does not contain any occurrence of p. Then, $(M, T) \underline{\leftrightarrow}_p (M', T')$ implies that (M, T) is Φ-complete iff (M', T') is Φ-complete.*

We end with a constraint on multi-pointed models, which is crucial for constructing transitive and Euclidean models. Intuitively, a multi-pointed model (M, T) is i-equivalent if the restriction of R_i to T is a complete graph and T is closed under the R_i relation.

Definition 21. Let $i \in \mathcal{A}$ and (M, T) be a multi-pointed model where $M = \langle S, R, V\rangle$. We say that (M, T) is i-*equivalent* if for all $t_1, t_2 \in T$, we have $t_1 R_i t_2$, and for every $s \in S$, if there exists $t \in T$ s.t. $sR_i t$ or $tR_i s$, then $s \in T$.

4.4. Forgetting in $\mathsf{K45_n}$, $\mathsf{KD45_n}$ and $\mathsf{S5_n}$

In this subsection, we generalize the forgetting results for satisfiable canonical formulas to $\mathsf{K45_n}$, $\mathsf{KD45_n}$ and $\mathsf{S5_n}$.

The following proposition says that any $\mathsf{K45_n}$ satisfiable canonical formula δ has the identical-children property: for any agent i and any i-child δ_i of δ, the lth-cut of δ's i-children is equal to the i-children of the $(l-1)$th-cut of δ_i.

Proposition 16. *Let δ be a $\mathsf{K45_n}$ satisfiable canonical formula where $dep(\delta) \geq 2$. Let $l \in \mathbb{N}$ s.t. $1 \leq l < dep(\delta)$. Then, for all $i \in \mathcal{A}$ and $\delta_i \in R_i(\delta)$, $(R_i(\delta))^{\downarrow l} = R_i(\delta_i^{\downarrow l-1})$.*

Now, we prove an important lemma, which is the multi-pointed extension of the back condition of the forgetting result for $\mathsf{K45_n}$.

Lemma 1 *(The $\mathsf{K45_n}$ lemma). Let δ be a $\mathsf{K45_n}$ satisfiable canonical formula where $dep(\delta) \geq 1$. Let (M', s') be a $\mathsf{K45_n}$ model of δ^p. Then for all $i \in \mathcal{A}$, there exists a multi-pointed $\mathsf{K45_n}$ model (M, T) that is i-equivalent, $R_i(\delta)$-complete and p-bisimilar to $(M', R_i'(s'))$.*

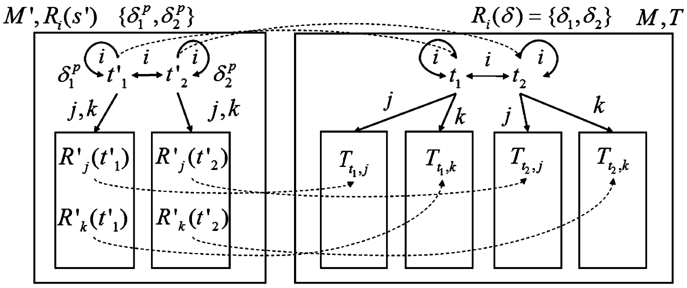

Fig. 6. Illustration for the proof of Lemma 1.

Proof sketch. Here, we only give the construction of the model (M, T) by induction on $dep(\delta)$. We will complete the proof in Appendix A.

Base case ($dep(\delta) = 1$): Suppose that $R_i(\delta)$ is a set of propositional formulas. We construct (M, T) and define the p-bisimulation ρ between (M, T) and $(M', R'_i(s'))$ as follows, and Fig. 6 illustrates the construction. Note that $t'_1 \in R'_i(s')$, $j, k \in \mathcal{A} \setminus \{i\}$, $R'_j(t'_1)$ and $R'_k(t'_1)$ may overlap so we put them in the same box. It is similar for t'_2. We initialize $S = \emptyset$ and $\rho = \emptyset$.

1. For all $t' \in R'_i(s')$ and $\delta_i \in R_i(\delta)$, if $M', t' \models \delta_i^p$, we create a world t s.t. $V(t) \models \delta_i$. Then we add it into S and T; and let $t\rho t'$. Finally, we let $t_1 R_i t_2$ for all $t_1, t_2 \in T$.
2. For $t \in T$ and $j \neq i$, we make a copy of $(M', R'_j(t'))$, denoted by $(M_{t,j}, T_{t,j})$ where $M_{t,j} = \langle S_{t,j}, R_{t,j}, V_{t,j}\rangle$ and t' is the original world of t. We connect t to all worlds of $T_{t,j}$ via j-edges, *i.e.*, let $tR_j u$ for $u \in T_{t,j}$. Let the p-bisimulation $\rho_{t,j}$ between $(M_{t,j}, T_{t,j})$ and $(M', R'_j(t'))$ be the set $\{(u, u') \mid u$ is the copy of $u'\}$, and expand ρ with $\rho_{t,j}$.

Induction step ($dep(\delta) > 1$): The construction of (M, T) is similar to that of the base case except the following. In Step 1, since δ_i is not propositional, we create a world t s.t. $V(t) \models w(\delta_i)$. In Step 2, we acquire $(M_{t,j}, T_{t,j})$ and $\rho_{t,j}$ by the induction hypothesis. Note that $(M_{t,j}, T_{t,j})$ is j-equivalent and $R_j(\delta_i)$-complete and $\rho_{t,j} : (M_{t,j}, T_{t,j}) \underline{\leftrightarrow}_p (M', R'_j(t'))$. $\square$

Now, we get the following theorem via Lemma 1.

Theorem 3 *(The* $\mathsf{K45_n}$ *theorem). Let δ be a* $\mathsf{K45_n}$ *satisfiable canonical formula. Then $kforget_{\mathsf{K45_n}}(\delta, p) \equiv \delta^p$.*

Proof sketch. The proof is the same as that of the basic theorem except Step 2 of the induction step. By the $\mathsf{K45_n}$ lemma, for every $i \in \mathcal{A}$, there exist (M_i, T_i) and $\rho_i : (M_i, T_i) \underline{\leftrightarrow}_p (M', R'_i(s'))$. Add a new copy of M_i into M, let $sR_i t$ for all $t \in T_i$, and expand ρ with ρ_i. It is obvious that $M, s \models \delta$. Similarly to the proof of Lemma 1, (M, s) is transitive, Euclidean, and p-bisimilar to (M', s'). $\square$

From the construction of the $\mathsf{K45_n}$ lemma and the $\mathsf{K45_n}$ theorem, if the given model M' is a $\mathsf{KD45_n}$ model, then we can acquire a $\mathsf{KD45_n}$ model M. Hence we can get

Theorem 4 *(The* $\mathsf{KD45_n}$ *theorem). Let δ be a* $\mathsf{KD45_n}$ *satisfiable canonical formula. Then $kforget_{\mathsf{KD45_n}}(\delta, p) \equiv \delta^p$.*

Now, we proceed to $\mathsf{S5_n}$. Since the logic $\mathsf{S5_n}$ is both $\mathsf{K45_n}$ and $\mathsf{T_n}$ logics, any $\mathsf{S5_n}$ satisfiable canonical formula has the reflexive and identical-children properties (Propositions 13 and 16), and hence the following property, called the *symmetric property*.

Proposition 17. *Let δ be an* $\mathsf{S5_n}$ *satisfiable canonical formula where $dep(\delta) \geq 2$. Let $l \in \mathbb{N}$ s.t. $2 \leq l \leq dep(\delta)$. Then, for all $i \in \mathcal{A}$ and $\delta_i \in R_i(\delta)$, $\delta^{\downarrow l} \in R_i(\delta_i^{\downarrow l-2})$.*

The models constructed in the proofs of the $\mathsf{K45_n}$ lemma and the $\mathsf{K45_n}$ theorem may not be reflexive. We can fix it via adding reflexive and symmetric edges.

Lemma 2 *(The* $\mathsf{S5_n}$ *lemma). Let δ be an* $\mathsf{S5_n}$ *satisfiable canonical formula where $dep(\delta) \geq 1$. Let (M', s') be an* $\mathsf{S5_n}$ *model of δ^p. Then for all $i \in \mathcal{A}$, there exists a multi-pointed* $\mathsf{S5_n}$ *model (M, T) that is i-equivalent, $R_i(\delta)$-complete and p-bisimilar to $(M', R'_i(s'))$.*

Proof sketch. Here, we only give the construction of the model (M, T). We will complete the proof in Appendix A.

As Fig. 7 shows, the construction of a proper model (M, T) is similar to that illustrated in the proof of the $\mathsf{K45_n}$ lemma except the following:

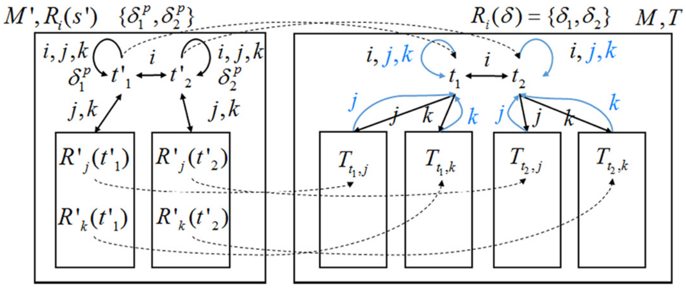

Fig. 7. Adding reflexive and symmetric edges.

- Each submodel $(M_{t,j}, T_{t,j})$ is not only a $\mathsf{K45_n}$ model, but also an $\mathsf{S5_n}$ model. This requirement can be fulfilled as follows: for the base case, $(M_{t,j}, T_{t,j})$ is a copy of $(M', R'_j(t'))$ and M' is an $\mathsf{S5_n}$ model; and for the induction step, we acquire an $\mathsf{S5_n}$ model $(M_{t,j}, T_{t,j})$ by the induction hypothesis of this lemma.
- After adding j-edges from the world t to all worlds of $T_{t,j}$, we add some edges for the models M so as to ensure that M is an $\mathsf{S5_n}$ model.
  1. Add reflexive edges for all worlds of T, *i.e.*, for every $t \in T$ and every $j \in \mathcal{A}$, we let tR_jt.
  2. Add symmetric edges from all worlds of $T_{t,j}$ to t where $t \in T$ and $j \in \mathcal{A} \setminus \{i\}$, *i.e.*, for every $u \in T_{t,j}$, we let uR_jt. □

Theorem 5 *(The $\mathsf{S5_n}$ theorem). Let δ be an $\mathsf{S5_n}$ satisfiable canonical formula. Then $kforget_{\mathsf{S5_n}}(\delta, p) \equiv \delta^p$.*

Proof sketch. The proof is the same as that of the $\mathsf{K45_n}$ theorem (Theorem 3) except the following:

1. In the induction step, by exploiting the $\mathsf{S5_n}$ lemma (Lemma 2), for every $i \in \mathcal{A}$, there exists an $\mathsf{S5_n}$ multi-pointed model (M_i, T_i) and $\rho_i : (M_i, T_i) \underline{\leftrightarrow}_p (M', R'_i(s'))$.
2. Similarly to the $\mathsf{S5_n}$ lemma, after the construction illustrated in Theorem 3, we add the reflexive edges for the actual world s, *i.e.*, for every $i \in \mathcal{A}$, we let sR_is; and we add the symmetric edges from all worlds of T_i to s, *i.e.*, for every world $t \in T_i$, we let tR_is.

As in the proof of Lemma 2, (M, s) is the required model. □

As a corollary of Propositions 6 and 10 and Theorems 2–5, we get

Corollary 3. *$\mathsf{T_n}$, $\mathsf{K45_n}$, $\mathsf{KD45_n}$ and $\mathsf{S5_n}$ are closed under forgetting.*

By Proposition 11 and Corollary 3, we can get the next corollary which settles the open problems regarding uniform interpolation in $\mathsf{K45_n}$ and $\mathsf{KD45_n}$.

Corollary 4. *$\mathsf{T_n}$, $\mathsf{K45_n}$, $\mathsf{KD45_n}$ and $\mathsf{S5_n}$ have uniform interpolation.*

5. Adding propositional common knowledge

A common feature of multi-agent systems is the presence of so-called background information. Such background information can be formalized in the modal language as common knowledge of propositional formulas. Given the relevance for practical applications of such propositional common knowledge, in this section, we generalize the results of the last section to include propositional common knowledge. In Subsection 5.1, we propose canonical formulas for propositional common knowledge. In Subsection 5.2, we consider forgetting in $\mathsf{K_nPC}$, $\mathsf{D_nPC}$ and $\mathsf{T_nPC}$. In Subsection 5.3, we handle forgetting in $\mathsf{K45_nPC}$, $\mathsf{KD45_nPC}$ and $\mathsf{S5_nPC}$.

5.1. Canonical formulas for propositional common knowledge

Similarly to the cover modality ∇_i, we use $\nabla\Phi$ to denote the formula $\mathbf{C}(\bigvee \Phi) \wedge (\bigwedge \hat{\mathbf{C}}\Phi)$. It is easy to prove the following proposition:

Proposition 18. *Let (M, s) be a Kripke model where $M = \langle S, R, V \rangle$ and $s \in S$. Then, $M, s \models \nabla\Phi$ iff the following conditions hold:*

Forth *For all $t \in R_{\mathcal{A}}(s)$, there is $\phi \in \Phi$ s.t. $M, t \models \phi$;*
Back *For all $\phi \in \Phi$, there is $t \in R_{\mathcal{A}}(s)$ s.t. $M, t \models \phi$.*

Moss [37] also defined canonical formulas for common knowledge: simply treat the common knowledge modality as just another knowledge modality. The definition makes use of the subscripted common knowledge operator $\mathbf{C}_G$ where $G \subseteq \mathcal{A}$. Similarly to the cover modality ∇, a cover modality ∇_G can be defined. The set D_k^P of canonical formulas for common knowledge is inductively defined as follows: D_0^P is the set of minterms of P; $D_{k+1}^P = \{\delta_0 \wedge \bigwedge_{i\in\mathcal{A}} \nabla_i \Phi_i \wedge \bigwedge_{G\subseteq\mathcal{A}} \nabla_G \Phi_G \mid \delta_0 \in D_0^P$ and $\Phi_i, \Phi_G \subseteq D_k^P\}$. Moss' definition of canonical formulas for common knowledge were used by Aucher and Belle [2] to give a new semantic account of multi-agent only-knowing.

In this paper, we only consider propositional common knowledge. In this subsection, we propose a definition of canonical formulas including propositional common knowledge. We call them pc-canonical formulas.

Definition 22 *(Pc-canonical formulas).* Let $P \subseteq \mathcal{P}$ be finite. We inductively define the set C_k^P as follows:

- $C_0^P = \{\delta_0 \wedge \nabla\Phi_{\mathcal{A}} \mid \delta_0 \in E_0^P$ and $\Phi_{\mathcal{A}} \subseteq E_0^P\}$;
- $C_{k+1}^P = \{\delta_0 \wedge (\bigwedge_{i\in\mathcal{A}} \nabla_i \Phi_i) \wedge \nabla\Phi_{\mathcal{A}} \mid \delta_0 \in E_0^P,\ \Phi_i \subseteq C_k^P$ and $\Phi_{\mathcal{A}} \subseteq E_0^P\}$.

We call each member of C_k^P a pc-canonical formula.

Let $\delta = \delta_0 \wedge \bigwedge_{i\in\mathcal{A}} \nabla_i \Phi_i \wedge \nabla\Phi_{\mathcal{A}}$. We define $w(\delta)$ and $R_i(\delta)$ as before; we denote $\Phi_{\mathcal{A}}$ by $R_{\mathcal{A}}(\delta)$, and call it the propositional descendants (p-descendants in short) of δ. Similarly to $E_k^P(\mathsf{L})$, we use $C_k^P(\mathsf{L})$ to denote the set of pc-canonical formulas satisfiable in a modal system L.

For $\phi \in \mathcal{L}_{\mathbf{PC}}^{\mathbf{K}}$, we introduce a modified definition of depth of ϕ, written in $dep'(\phi)$. It is the same as the classical definition of depth except that we ignore the common knowledge operator, *i.e.*, $dep'(\mathbf{C}\phi) = dep'(\phi)$.

Similarly to Propositions 3–6 and 14, we can get the following propositions for pc-canonical formulas.

Proposition 19. *Consider the context of a modal system L. Let $\delta \in C_k^P(\mathsf{L})$ where $k \in \mathbb{N}$ and $P \subseteq \mathcal{P}$ be finite. Let $\phi \in \mathcal{L}_{\mathbf{PC}}^{\mathbf{K}}$ s.t. $dep'(\phi) \le k$ and $\mathcal{P}(\phi) \subseteq P$. Then either $\delta \models \phi$ or $\delta \models \neg\phi$.*

Moreover, the following proposition gives us an algorithm to check if $\delta \models \phi$.

Proposition 20. *Consider the context of a modal system L. Let $\delta \in C_k^P(\mathsf{L})$ where $k \in \mathbb{N}$ and $P \subseteq \mathcal{P}$ is finite. Let $\phi \in \mathcal{L}_{\mathbf{PC}}^{\mathbf{K}}$ s.t. $dep'(\phi) \le k$ and $\mathcal{P}(\phi) \subseteq P$. Then we can check if $\delta \models \phi$ recursively as follows:*

- *$\delta \models p$ iff p appears positively in $w(\delta)$;*
- *$\delta \models \neg\phi$ iff $\delta \not\models \phi$;*
- *$\delta \models \phi \wedge \psi$ iff $\delta \models \phi$ and $\delta \models \psi$;*
- *$\delta \models \phi \vee \psi$ iff $\delta \models \phi$ or $\delta \models \psi$;*
- *$\delta \models \mathbf{K}_i\phi$ iff for all $\delta' \in R_i(\delta)$, $\delta' \models \phi$.*
- *$\delta \models C\phi$ iff for all $\delta' \in R_{\mathcal{A}}(\delta)$, $\delta' \models \phi$.*

Proposition 21. *Let (M, s) be a model and $k \in \mathbb{N}$. Let $P \subseteq \mathcal{P}$ be finite. Then, there exists a unique $\delta \in C_k^P$ s.t. $M, s \models \delta$.*

Proposition 22. *Let (M, T) be a multi-pointed model and $k \in \mathbb{N}$. Let $P \subseteq \mathcal{P}$ be finite. Then, there exists a unique set $\Phi \subseteq C_k^P$ s.t. (M, T) is Φ-complete.*

Proposition 23. *Consider the context of a modal system L. Let $\phi \in \mathcal{L}_{\mathbf{PC}}^{\mathbf{K}}$, $k \ge dep'(\phi)$ and $P = \mathcal{P}(\phi)$. Then there exists a unique set $\Phi \subseteq C_k^P(\mathsf{L})$ s.t. $\phi \equiv \bigvee \Phi$.*

Similarly to Proposition 6, for the modal systems listed in Table 1, there is an algorithm to construct Φ in the above proposition. Firstly, we construct C_k^P. Then we remove from it those formulas unsatisfiable in L: for all the modal systems in Table 1 with common knowledge, satisfiability is decidable [21]. Finally, by using Proposition 20, we remove from $C_k^P(\mathsf{L})$ those formulas which entail $\neg\phi$.

Now we extend the projection operations to pc-canonical formulas. Intuitively, given a pc-canonical formula δ, $\delta^{\downarrow}$ is acquired by pruning the leaves of δ except maintaining the propositional common knowledge.

Definition 23. Let $P \subseteq \mathcal{P}$ be finite. Let $k \in \mathbb{N}$ and $\delta \in C_k^P$. Then, $\delta^{\downarrow}$ is inductively defined as follows:

$$\delta^{\downarrow} = \begin{cases} \delta, & \text{if } k = 0; \\ w(\delta) \wedge \nabla R_A(\delta), & \text{if } k = 1; \\ w(\delta) \wedge \bigwedge_{i\in\mathcal{A}} \nabla_i (R_i(\delta))^{\downarrow} \wedge \nabla R_A(\delta), & \text{otherwise.} \end{cases}$$

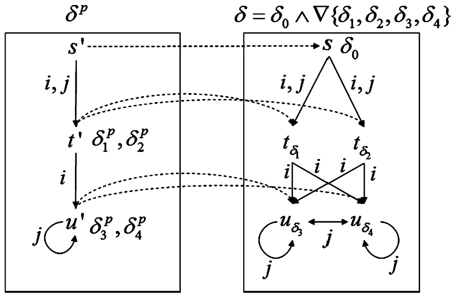

Fig. 8. Illustration for the proof of Theorem 6.

For a pc-canonical formula δ, the definition of $\delta^{\downarrow l}$ is the same as Definition 8. Similarly to Proposition 7, the two operations R_i and $\downarrow l$ are commutative for pc-canonical formulas whose depth is greater than l.

Proposition 24. *Let δ be a pc-canonical formula and $l < dep'(\delta)$. Then for all $i \in \mathcal{A}$, we have $R_i(\delta^{\downarrow l}) = (R_i(\delta))^{\downarrow l}$.*

Proposition 12 can be extended to pc-canonical formulas. Given a pc-canonical formula δ, literal elimination leads to a formula which is entailed by δ.

Proposition 25. *Let δ be a pc-canonical formula and p an atom. Then, $\delta \models \delta^p$.*

5.2. Forgetting in $\mathsf{K_n PC}$, $\mathsf{D_n PC}$ and $\mathsf{T_n PC}$

In this subsection, we prove that forgetting via literal elimination works on satisfiable pc-canonical formulas in the logics of $\mathsf{K_n PC}$, $\mathsf{D_n PC}$ and $\mathsf{T_n PC}$. The proofs of the induction steps are similar to those without common knowledge. However, the proofs of the base cases are not straightforward. We first analyze properties of satisfiable pc-canonical formulas.

The following proposition says that any satisfiable pc-canonical formula has the transitive closure property: any p-descendant is either the world of an i-child or a p-descendant of an i-child for some agent i.

Proposition 26. *Let δ be a satisfiable pc-canonical formula where $dep'(\delta) \geq 1$. Then $R_{\mathcal{A}}(\delta) = \bigcup_{i \in \mathcal{A}} \bigcup_{\delta_i \in R_i(\delta)} [\{w(\delta_i)\} \cup R_{\mathcal{A}}(\delta_i)]$.*

Theorem 6 *(The $\mathsf{K_n PC}$/$\mathsf{D_n PC}$ theorem). Let L be $\mathsf{K_n PC}$ or $\mathsf{D_n PC}$, and δ an L-satisfiable pc-canonical formula. Then $kforget_{\mathsf{L}}(\delta, p) \equiv \delta^p$.*

Proof. Base case: We construct M and define a relation ρ between the worlds of M and M' as follows. Fig. 8 illustrates the construction.

1. Create a new world s, and then let $s \rho s'$ and $V(s) \models w(\delta)$.
2. For every $t' \in R'_{\mathcal{A}}(s')$ and every $\delta_{\mathcal{A}} \in R_{\mathcal{A}}(\delta)$, if $M', t' \models \delta^p_{\mathcal{A}}$, we create a world t s.t. $V(t) \models \delta_{\mathcal{A}}$, and let $t \rho t'$.
3. For every $i \in \mathcal{A}$, we let $R_i = \{(s, t) \mid s' R'_i t' \text{ and } t \in S \setminus \{s\}\} \cup \{(t, u) \mid t' R'_i u' \text{ and } t, u \in S \setminus \{s\}\}$, where t and u are copies of t' and u' respectively.

Obviously, $M, s \models \delta$ and $(M, s) \underline{\leftrightarrow}_p (M', s')$.

Induction step: The model construction is similar to that of the induction step of the basic theorem (Theorem 1). Obviously, $M, s \models w(\delta) \wedge \bigwedge_{i \in \mathcal{A}} \nabla_i R_i(\delta)$ and $(M, s) \underline{\leftrightarrow}_p (M', s')$. It remains to verify $M, s \models \nabla R_{\mathcal{A}}(\delta)$. Here, we only prove for every $t \in R_{\mathcal{A}}(s)$, there exists $\delta_{\mathcal{A}} \in R_{\mathcal{A}}(\delta)$ s.t. $M, t \models \delta_{\mathcal{A}}$. A world t reachable from s is either an i-child of s, or a world reachable from an i-child of s for some $i \in \mathcal{A}$. If $t \in R_i(s)$, then there exists $\delta_i \in R_i(\delta)$ s.t. $M, t \models \delta_i$, so $M, t \models w(\delta_i)$. By the transitive closure property (Proposition 26), $w(\delta_i) \in R_{\mathcal{A}}(\delta)$. If $t \in R_{\mathcal{A}}(u)$ where $u \in R_i(s)$, then there exists $\delta_i \in R_i(\delta)$ s.t. $M, u \models \delta_i$; and there exists $\delta_{\mathcal{A}} \in R_{\mathcal{A}}(\delta_i)$ s.t. $M, t \models \delta_{\mathcal{A}}$. By Proposition 26, $\delta_{\mathcal{A}} \in R_{\mathcal{A}}(\delta_i) \subseteq R_{\mathcal{A}}(\delta)$. $\square$

The general reflexive property of $\mathsf{T_n}$ satisfiable canonical formulas (Proposition 13) can be extended to $\mathsf{T_n PC}$:

Proposition 27. *Let δ be a $\mathsf{T_n PC}$ satisfiable pc-canonical formula where $dep'(\delta) \geq 1$. Let $l \in \mathbb{N}$ s.t. $1 \leq l \leq dep'(\delta)$. Then, for every $i \in \mathcal{A}$, we have $\delta^{\downarrow l} \in R_i(\delta^{\downarrow l-1})$.*

Similarly, any $\mathsf{T_n PC}$-satisfiable pc-canonical formula has the pc-reflexive property, which says that the world of δ belongs to the set of p-descendants of δ.

Proposition 28. *Let δ be a* $\mathsf{T_nPC}$ *satisfiable pc-canonical formula. Then, we have $w(\delta) \in R_{\mathcal{A}}(\delta)$.*

With the above properties of $\mathsf{T_nPC}$-satisfiable pc-canonical formulas, we can prove that $\mathsf{T_nPC}$ is closed under forgetting.

Theorem 7. *Let δ be a* $\mathsf{T_nPC}$ *satisfiable pc-canonical formula. Then $kforget_{\mathsf{T_nPC}}(\delta, p) \equiv \delta^p$.*

Proof sketch. The proof is the same as that of the $\mathsf{K_nPC}/\mathsf{D_nPC}$ theorem except the following. As in the proof of the $\mathsf{T_n}$ theorem, in the induction step, we let sR_is for each agent i. Obviously, (M, s) is reflexive and p-bisimilar to (M', s'). As in the proof of the $\mathsf{T_n}$ theorem, we get $M, s \models w(\delta) \wedge \bigwedge_{i\in\mathcal{A}} \nabla_i R_i(\delta)$ via the general reflexive property. As in the proof of the $\mathsf{K_nPC}/\mathsf{D_nPC}$ theorem, we get $M, s \models \nabla R_{\mathcal{A}}(\delta)$ via the transitive closure and pc-reflexive properties. □

5.3. Forgetting in $\mathsf{K45_nPC}$, $\mathsf{KD45_nPC}$ and $\mathsf{S5_nPC}$

Finally, we extend the forgetting results to the logics of $\mathsf{K45_nPC}$, $\mathsf{KD45_nPC}$ and $\mathsf{S5_nPC}$.

The identical-children property of $\mathsf{K45_n}$ satisfiable canonical formulas (Proposition 16) can be extended to $\mathsf{K45_nPC}$:

Proposition 29. *Let δ be a* $\mathsf{K45_nPC}$ *satisfiable pc-canonical formula where $dep'(\delta) \geq 2$. Let $l \in \mathbb{N}$ s.t. $1 \leq l < dep'(\delta)$. Then, for all $i \in \mathcal{A}$ and $\delta_i \in R_i(\delta)$, $(R_i(\delta))^{\downarrow l} = R_i(\delta_i^{\downarrow l-1})$.*

Similarly, any $\mathsf{K45_nPC}$-satisfiable pc-canonical formula δ has the following properties. The first property means that for every agent i and every i-child δ_i of δ, there is an i-child δ'_i of δ such that any p-descendant of δ_i is either the world of δ'_i or a p-descendant of δ'_i.

Proposition 30. *Let δ be a* $\mathsf{K45_nPC}$ *satisfiable pc-canonical formula where $dep'(\delta) \geq 1$. Then, for all $i \in \mathcal{A}$ and $\delta_i \in R_i(\delta)$, we have $R_{\mathcal{A}}(\delta_i) = \bigcup_{\delta'_i \in R_i(\delta)} [\{w(\delta'_i)\} \cup R_{\mathcal{A}}(\delta'_i)]$.*

The second property means that for every agent i and every i-child δ_i and δ'_i of δ, the set of p-descendants of δ_i and that of δ_i are equal.

Proposition 31. *Let δ be a* $\mathsf{K45_nPC}$ *satisfiable pc-canonical formula where $dep'(\delta) \geq 1$. Then, for all $i \in \mathcal{A}$ and $\delta_i, \delta'_i \in R_i(\delta)$, we have $R_{\mathcal{A}}(\delta_i) = R_{\mathcal{A}}(\delta'_i)$.*

Now, we extend the $\mathsf{K45_n}$ lemma to $\mathsf{K45_nPC}$ and $\mathsf{KD45_nPC}$.

Lemma 3 *(The $\mathsf{K45_nPC}/\mathsf{KD45_nPC}$ lemma). Let L be $\mathsf{K45_nPC}$ or $\mathsf{KD45_nPC}$, and $i \in \mathcal{A}$. Let δ be an L-satisfiable pc-canonical formula where $dep'(\delta) \geq 1$. Let (M', s') be an L-model of δ^p. Then there exists a multi-pointed L-model (M, T) that is i-equivalent, $R_i(\delta)$-complete and p-bisimilar to $(M', R'_i(s'))$.*

Proof. The proof combines the proofs of the $\mathsf{K45_n}$ lemma and the $\mathsf{K_nPC}/\mathsf{D_nPC}$ theorem. The main construction is the same as that of the $\mathsf{K45_n}$ lemma except the base case as follows: In Step 1, since δ_i is not propositional, we create a world t s.t. $V(t) \models w(\delta_i)$. In Step 2, we cannot let $(M_{t,j}, T_{t,j})$ be a copy of $(M', R'_j(t'))$ due to the presence of common knowledge. As in the proof of the $\mathsf{K_nPC}/\mathsf{D_nPC}$ theorem, we construct the worlds of $M_{t,j}$ and the p-bisimulation $\rho_{t,j}$ between the worlds of $M_{t,j}$ and M' by possibly splitting a world into two copies according to the set $R_{\mathcal{A}}(\delta_i)$. Then, we let $T_{t,j} = \{u \mid u' \in R'_j(t')\}$, and let $R_k = \{(u, v) \mid u' R'_k v'\}$ for $k \in \mathcal{A}$, where t, u and v are copies of t', u' and v' respectively. By the proofs of the $\mathsf{K45_n}$ lemma and the $\mathsf{K_nPC}/\mathsf{D_nPC}$ theorem, we get that (M, T) is transitive, Euclidean, i-equivalent and p-bisimilar to $(M', R'_i(s'))$. It remains to prove that (M, T) is $R_i(\delta)$-complete. Here, we only prove for every $t \in T$, there exists $\delta_i \in R_i(\delta)$ s.t. $M, t \models \delta_i$. The back condition can be similarly proved. We prove by induction on $dep'(\delta)$.

Base case ($dep'(\delta) = 1$): By the construction, there exist $t' \in S'$ and $\delta_i \in R_i(\delta)$ s.t. $t\rho t'$, $M', t' \models \delta_i^p$ and $M, t \models w(\delta_i)$. Now, we prove that $M, t \models \nabla R_{\mathcal{A}}(\delta_i)$. Here we only prove that for every $v \in R_{\mathcal{A}}(t)$, there exists $\delta_{\mathcal{A}} \in R_{\mathcal{A}}(\delta_i)$ s.t. $M, v \models \delta_{\mathcal{A}}$.

1. $v \in T$: By the construction, there exist $v' \in S'$ and $\delta'_i \in R_i(\delta)$ s.t. $v\rho v'$, $M', v' \models \delta_i'^p$ and $V(v) \models w(\delta'_i)$. By Proposition 30, we get that $w(\delta'_i) \in R_{\mathcal{A}}(\delta_i)$. So $w(\delta'_i)$ is the desired $\delta_{\mathcal{A}}$.
2. $v \notin T$: Since R_i is transitive and Euclidean and $v \notin T$, v is a descendant of u where $u \in T$ and $v \notin R_i(u)$. By the idea of possibly splitting the worlds into two copies, there exist $v' \in S'$ and $\delta_{\mathcal{A}} \in R_{\mathcal{A}}(\delta_i)$ s.t. $v\rho v'$, $M', v' \models \delta_{\mathcal{A}}^p$, and $V(v) \models \delta_{\mathcal{A}}$. Since $\delta_{\mathcal{A}}$ is propositional, $M, v \models \delta_{\mathcal{A}}$.

Induction step ($dep'(\delta) > 1$): Similarly to the base case, there exist $t' \in S'$ and $\delta_i \in R_i(\delta)$ s.t. $t\rho t'$, $M', t' \models \delta_i^p$ and $V(t) \models w(\delta_i)$. As in the proof of the $\mathsf{K45_n}$ lemma, we get $M, t \models \bigwedge_{j\in\mathcal{A}} \nabla_j R_j(\delta_i)$ via the general identical-children property (Proposition 29).

Now, we prove $M, t \models \nabla R_{\mathcal{A}}(\delta_i)$. Here we only prove that for every $v \in R_{\mathcal{A}}(t)$, there exists $\eta \in R_{\mathcal{A}}(\delta)$ s.t. $M, v \models \eta$. The proof of the situation that $v \in T$ is the same as that in the base case. It remains to verify the situation that $v \notin T$.

Since $v \notin T$ and R_i is transitive and Euclidean, v is a descendant of u where $u \in T$. By the construction, there exists $(M_{u,j}, T_{u,j})$ that is $R_j(\delta'_i)$-complete. Then, there exists $\delta'_i \in R_i(\delta)$ s.t. $M_{u,j}, u \models \delta'_i$. By the semantics of the ∇ modality, there exists $\delta_{\mathcal{A}} \in R_{\mathcal{A}}(\delta'_i)$ s.t. $M_{u,j}, v \models \delta_{\mathcal{A}}$. Since $\delta_{\mathcal{A}}$ is propositional, $M, v \models \delta_{\mathcal{A}}$. By Propositions 30 and 31, we get $\delta_{\mathcal{A}} \in R_{\mathcal{A}}(\delta'_i) = R_{\mathcal{A}}(\delta_i)$. □

Theorem 8 *(The* $\mathsf{K45_nPC/KD45_nPC}$ *theorem). Let* L *be* $\mathsf{K45_nPC}$ *or* $\mathsf{KD45_nPC}$*, and* δ *an* L*-satisfiable pc-canonical formula. Then* $kforget_{\mathsf{L}}(\delta, p) \equiv \delta^p$.

Proof sketch. In the base case, we can construct a required model according to the proof of the $\mathsf{K_nPC/D_nPC}$ theorem. The proof of the induction step via Lemma 3 is the same as that of the $\mathsf{K45_n}$ theorem except we acquire the submodels (M_i, T_i) for $i \in \mathcal{A}$ by the $\mathsf{K45_nPC}$ lemma, and verify $M, s \models R_{\mathcal{A}}(\delta)$ by utilizing the proof of the $\mathsf{K_nPC/D_nPC}$ theorem and the $\mathsf{K45_nPC}$ lemma. □

Each model of $\mathsf{S5_nPC}$ is reflexive, transitive and Euclidean, so any $\mathsf{S5_nPC}$ satisfiable pc-canonical formula has the general reflexive, pc-reflexive and general identical children properties.

Recall the model construction method for $\mathsf{S5_n}$: based on the models constructed in the proofs of the $\mathsf{K45_n}$ lemma and the $\mathsf{K45_n}$ theorem, we add reflexive and symmetric edges. Similarly, Lemma 3 and Theorem 8 can be extended to $\mathsf{S5_nPC}$.

Lemma 4 *(The* $\mathsf{S5_nPC}$ *lemma). Let* δ *be an* $\mathsf{S5_nPC}$*-satisfiable pc-canonical formula where* $dep'(\delta) \geq 1$*. Let* (M', s') *be an* $\mathsf{S5_nPC}$ *model of* δ^p*. Then for all* $i \in \mathcal{A}$*, there exists a multi-pointed* $\mathsf{S5_nPC}$ *model* (M, T) *that is* i*-equivalent,* $R_i(\delta)$*-complete and* p*-bisimilar to* $(M', R'_i(s'))$.

Proof. The construction of a proper model is the same as that of the $\mathsf{S5_n}$ lemma (Lemma 2) except we first acquire the multi-pointed $\mathsf{K45_nPC}$-model via the $\mathsf{K45_nPC}$ lemma (Lemma 3). As in the proof of Lemma 2, (M, T) is reflexive, Euclidean, i-equivalent and p-bisimilar to $(M', R'_i(s'))$. It remains to prove that it is $R_i(\delta)$-complete.

To distinguish the models before and after adding reflexive and symmetric edges, we let (M^*, T^*) be the model acquired via Lemma 3 where $M^* = \langle S^*, R^*, V^* \rangle$, and (M, T) be the model after adding edges where $M = \langle S, R, V \rangle$, and ρ be the p-bisimulation between (M^*, s^*) and (M', s'). By Proposition 15, to prove (M, T) is $R_i(\delta)$-complete, it suffices to prove that $(M, T) \underline{\leftrightarrow} (M^*, T^*)$. We construct the bisimulation ρ^* between S and S^* as follows:

1. Initialize $\rho^* = \{(s, s^*) \mid s \text{ is a copy of } s^*\}$.
2. Since we add reflexive and symmetric edges in Steps 2 and 3 of the construction illustrated in the $\mathsf{S5_n}$ lemma respectively, the forth condition of ρ^* may not hold. For example, suppose that tR_jt and $t\rho t^*$ where $t \in T$ and $t^* \in T^*$. It is possible that there does not exist $u^* \in S^*$ s.t. $t^* R^*_j u^*$ and $t\rho^* u^*$. So it is necessary to add some pairs of worlds of M and M^* into the bisimulation ρ^* for every $t \in T$. Let t^* be the original world of t in S^*.
   Base case ($dep'(\delta) = 1$**):** By the construction, there exist $t' \in S'$ and $\delta_i \in R_i(\delta)$ s.t. $t^* \rho t'$, $M', t' \models \delta^p_i$ and $M^*, t^* \models \delta_i$. Since M' is reflexive, $t' \in R'_j(t')$ for any agent $j \neq i$. By the pc-reflexive property (Proposition 28), we have $w(\delta_i) \in R_{\mathcal{A}}(\delta_i)$. By the construction, there exists $u^* \in S^*$ s.t. $u^* \in R^*_j(t^*)$ and $M^*, u^* \models w(\delta_i)$. We let $\rho^*(t, u^*)$.
   Induction step ($dep'(\delta) > 1$**):** Similarly to the base case, there exists $t' \in S'$ and $\delta_i \in R_i(\delta)$ s.t. $t^* \rho t'$, $M', t' \models \delta^p_i$, $M^*, t^* \models \delta_i$, and $t' \in R'_j(t')$ for any agent $j \neq i$. By the general reflexive property (Proposition 27), $\delta^{\downarrow}_i \in R_j(\delta_i)$. By the construction of models of the induction step, there exists $(M^*_{t^*,j}, T^*_{t^*,j})$ that is $R_j(\delta_i)$-complete. So there exists $u^* \in T^*_{t^*,j}$ s.t. $M^*, u^* \models \delta^{\downarrow}_i$. We let $\rho^*(t, u^*)$.

It is easy to check that most cases of the forth condition and the atom and back conditions hold. Here, we only prove the following cases of the forth condition due to adding reflexive and symmetric edges. Suppose that $u, v \in S$ and $u^* \in S^*$ s.t. uR_jv and $(u, u^*) \in \rho^*$.

1. $u, v \in T$ and $j \neq i$: Since we only add a j-edge from u to itself, we get $u = v$. By the construction of ρ^*, there exists $w^* \in S^*$ s.t. $w^* \in R^*_j(u^*)$ and $v\rho^* w^*$.
2. $u \notin T$ and $v \in T$: In Step 3, we only add a j-edge from u to v where $j \neq i$ and $u \in T_{v,j}$. Let v^* be the original world of v. Similarly to Case 1, there exists $w^* \in R^*_j(v^*)$ s.t. $v\rho w^*$. Obviously, $u^* \in R^*_j(v^*)$. Since M^* is Euclidean, $w^* \in R^*_j(u^*)$. □

Theorem 9 *(The* $\mathsf{S5_nPC}$ *theorem). Let* δ *be an* $\mathsf{S5_nPC}$ *satisfiable pc-canonical formula. Then* $kforget_{\mathsf{S5_nPC}}(\delta, p) \equiv \delta^p$.

Proof sketch. The proof is the same as that of the $\mathsf{S5_n}$ theorem (Theorem 5) except the following: In the induction step, for every $i \in \mathcal{A}$, we acquire a multi-pointed model (M_i, T_i) and $\rho_i : (M_i, T_i) \underline{\leftrightarrow}_p (M', R'_i(s'))$ by exploiting the $\mathsf{S5_nPC}$ lemma (Lemma 2). As in the proof of Lemma 4, (M, s) is the required model. □

Finally, by Propositions 23 and Theorems 6–9, we get

Corollary 5. $\mathsf{K_n PC}$, $\mathsf{D_n PC}$, $\mathsf{T_n PC}$, $\mathsf{K45_n PC}$, $\mathsf{KD45_n PC}$ *and* $\mathsf{S5_n PC}$ *are closed under forgetting, and hence have uniform interpolation.*

6. Related work

In addition to the works already discussed at length in the introductory section, let us now present some significant technical details in view of the specific results obtained in our own contribution. We will pay special attention to the *definitions* of forgetting, to the different methods for *computing* forgetting, and to the difference between canonical formulas and *disjunctive formulas*.

Definition of forgetting varies in terms of the following aspects: whether the result of forgetting is finitely representable, whether the definition is model-theoretic or via postulates, and how strong the definition is. Model-theoretic definitions of forgetting are based on various similarity relations between models. Basic requirements for forgetting, as expressed by the four forgetting postulates of Zhang and Zhou [48], are the following: the result of forgetting should be weaker than the original theory/formula, it should be irrelevant to the forgotten symbol, and it should have the same set of logical consequences irrelevant to the forgotten symbol. Lin and Reiter's definition of forgetting in first-order logic does not require the result of forgetting to be finitely representable, it is a model-theoretic definition based on the identity relation between two models, and hence it results in a strong concept of forgetting. Zhang and Zhou [49] proposed the notion of weak forgetting for first-order logic using their four postulates. They showed that the result of weak forgetting a predicate P from a theory T is equivalent to the set of first-order logical consequences of T irrelevant to P. They gave a model-theoretic characterization of their definition: the models of the result of weak forgetting P from T are exactly those structures M' which are elementarily equivalent to a model M of T with exception on P, *i.e.*, M and M' agree on every sentence irrelevant to P. Obviously, the elementary equivalence relation is weaker than the identity relation. Our definition of forgetting in multi-agent modal logics requires the result of forgetting to be finitely representable, it is a model-theoretic definition based on the bisimulation relation between two Kripke models, and it satisfies the four postulates of Zhang and Zhou. Hence our definition of forgetting is weaker than the direct application of Lin and Reiter's definition to modal logics, and stronger than Zhang and Zhou's postulate-based definition.

Two common approaches for computing forgetting are as follows: the first is based on conjunctive normal forms and uses resolution; the second is based on disjunctive normal forms and uses literal elimination. These issues were already summarily discussed in the introduction. Let us now discuss both methods in more detail.

Ackermann [1] proposed a resolution method for second-order quantifier elimination (SOQE), *i.e.*, forgetting in first-order logic. As a refinement of Ackermann's resolution method, Gabbay and Ohlbach [15] developed the SCAN algorithm for SOQE. For the expressive description logic $\mathcal{ALC}$, Koopmann and Schmidt [26] presented a resolution-based method for computing forgetting for $\mathcal{ALC}$ ontologies. The method was further developed in [51,52]. For the modal logic K, Herzig and Mengin [23] proposed a resolution-based method for computing forgetting for formulas in conjunctive normal form.

Janin and Walukiewicz [24] introduced the so-called disjunctive formulas for the μ-calculus, and showed that every formula in the μ-calculus is equivalent to a disjunctive formula. The disjunctive formulas for $\mathsf{K_n}$ are defined as follows: 1. If ϕ_1 and ϕ_2 are disjunctive formulas, then $\phi_1 \vee \phi_2$ is a disjunctive formula; 2. If δ_0 is a consistent conjunction of literals, and for each $i \in \mathcal{B} \subseteq \mathcal{A}$, Φ_i is a finite set of disjunctive formulas, then $\delta_0 \wedge \bigwedge_{i\in\mathcal{B}} \nabla_i \Phi_i$ is a disjunctive formula. D'Agostino and Lenzi [8] used literal elimination to compute forgetting for disjunctive formulas, and ten Cate et al. [6] presented a single-exponential algorithm for computing forgetting for $\mathcal{ALC}$ concepts ($\mathcal{ALC}$ corresponds to $\mathsf{K_n}$): first put the concept into a form corresponding to disjunctive formulas, and then do literal elimination. The conversion to disjunctive form might involve a single exponential blowup.

The disjunctive normal formulas we use in this paper are disjunctions of satisfiable canonical formulas. Every modal formula is equivalent to such a formula of possibly non-elementary complexity. The canonical formulas of Moss [37] are similar to the normal forms of propositional modal logics by Fine [13]. However, both Moss and Fine used canonical formulas to construct finite Kripke models in order to prove completeness and decidability results for standard modal systems.

7. Conclusions and future work

In this paper, we have studied forgetting in multi-agent modal logics. We adopted the semantic definition of existential bisimulation quantifiers as that of forgetting. To investigate whether a modal system is closed under forgetting, we resorted to canonical formulas introduced by Moss. An arbitrary modal formula is equivalent to the disjunction of a unique set of satisfiable canonical formulas, and there is an algorithm to construct this set for any modal system whose satisfiability is decidable. We showed that for the logics of $\mathsf{K_n}$, $\mathsf{D_n}$, $\mathsf{T_n}$, $\mathsf{K45_n}$, $\mathsf{KD45_n}$, and $\mathsf{S5_n}$, the result of forgetting an atom from a satisfiable canonical formula can be obtained by the simple method of literal elimination, *i.e.*, substituting the literals of the atom with $\top$. Therefore, with a uniform and constructive proof method, we showed that all these modal systems are closed under forgetting and hence have uniform interpolation. This also settles the open problems whether $\mathsf{K45_n}$ and $\mathsf{KD45_n}$ have uniform interpolation. Moreover, we generalized the above results to include propositional common knowledge. The results are summarized in Table 2, which contains an additional row for logics with (unrestricted) common knowledge. In the table,

Table 2
Summary of forgetting in multi-agent modal logics.

L	K	D	T	K4	S4	K45	KD45	S5
$\mathcal{L}_n^{\mathbf{K}}$	✓	✓	✓	×	×	✓	✓	✓
$\mathcal{L}_{\mathbf{PC}}^{\mathbf{K}}$	✓	✓	✓	×	×	✓	✓	✓
$\mathcal{L}_{\mathbf{C}}^{\mathbf{K}}$	×	?	?	×	×	?	?	?

the symbol "✓" denotes that the modal system is closed under forgetting, "×" denotes that it is not, and "?" means that the issue remains open.

We give a model-theoretic proof that literal elimination generates the result of forgetting for satisfiable canonical formulas: given a model satisfying the formula obtained from a satisfiable canonical formula via literal elimination, we construct a model satisfying the original formula and p-similar to the given model. To this end, we analyze properties of satisfiable canonical formulas in different logics, use multi-pointed models to gain flexibility in model construction, and resort to making multiple copies of a world or a sub-model. In $\mathsf{K_n}$ and $\mathsf{D_n}$, we directly construct a desired model by induction on the modal depth of the canonical formula. In $\mathsf{T_n}$, we first do the above construction, and then add reflexive edges for the root world. In $\mathsf{K45_n}$, we first construct a proper multi-pointed model (M_i, T_i) for each agent i, and then add an i-edge from the root world to each world of T_i. In $\mathsf{S5_n}$, we first do the construction of $\mathsf{K45_n}$, and then add a reflexive i-edge for the root world and add an symmetric i-edge from each world of T_i to the root world. Satisfiable canonical formulas of different systems have different properties, and we use these properties to prove that the constructed model satisfies the original canonical formula. For example, for $\mathsf{T_n}$ (resp. $\mathsf{K45_n}$), we make use of the reflexive (resp. identical-children) property. When we add propositional common knowledge, in the base case, we make multiple copies of a world if necessary.

We have focused on the issue whether a modal system is closed under forgetting. Our results suggest an algorithm for computing forgetting for each modal system we have considered: to forget an atom from an arbitrary modal formula, first transform the formula to a disjunction of canonical formulas, then remove all the ones unsatisfiable in the given logic, and lastly substitute any literal of the atom with $\top$. However, the size of a canonical formula is non-elementary. So the proposed syntactic method of forgetting is obviously an unpractical one. Thus one topic for future research is to investigate more efficient approaches for computing forgetting. A possible approach is to identify more succinct disjunctive normal forms for different modal logics for which forgetting can be computed in polynomial time. For example, disjunctive formulas [24] are such normal forms for $\mathsf{K_n}$; candidates for $\mathsf{KD45_n}$ are *alternating disjunctive formulas* [20], which are disjunctive formulas such that modal operators of an agent do not directly occur inside those of the same agent. Another possibility is to identify conjunctive normal forms for different modal logics and use resolution to compute forgetting.

We have only considered common knowledge of propositional formulas. So another topic for future research is to explore forgetting for more general cases of common knowledge. Moss [37] defined canonical formulas for arbitrary common knowledge. As neither KC nor K4C has uniform interpolation [39], we might start with exploring under what conditions literal elimination produces the correct result of forgetting for satisfiable canonical formulas for KC and K4C.

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Appendix A. Proofs

In this section, we give proofs for results in the paper in the order they appear in the paper with one exception: Proposition 14 is a generalization of Proposition 5, so we prove them together.

Proposition 1. *Let (M, s) be a Kripke model where $M = \langle S, R, V\rangle$ and $s \in S$. Then, $M, s \models \nabla_i \Phi$ iff the following conditions hold:*

Forth *For all $t \in R_i(s)$, there is $\phi \in \Phi$ s.t. $M, t \models \phi$;*
Back *For all $\phi \in \Phi$, there is $t \in R_i(s)$ s.t. $M, t \models \phi$.*

Proof. Firstly, by Definition 5, we get that $\nabla_i \Phi \equiv \mathbf{K}_i(\bigvee \Phi) \wedge (\bigwedge \hat{\mathbf{K}}_i \Phi)$.

($\Rightarrow$) Let (M, s) be a model of $\nabla_i \Phi$. Since $M, s \models \mathbf{K}_i(\bigvee \Phi)$, we get that for all $t \in R_i(s)$, $M, t \models \bigvee \Phi$. By the semantics of disjunction, for all $t \in R_i(s)$, there is $\phi \in \Phi$ s.t. $M, t \models \phi$. In addition, $M, s \models \bigwedge \hat{\mathbf{K}}_i \Phi$. It follows that for all $\phi \in \Phi$, there is $t \in R_i(s)$ s.t. $M, t \models \phi$.

($\Leftarrow$) Let (M, s) be a model satisfying the RHS condition. Since for all $t \in R_i(s)$, there is $\phi \in \Phi$ s.t. $M, t \models \phi$, we have for all $t \in R_i(s)$, $M, t \models \bigvee \Phi$. Then, we get that $M, s \models \mathbf{K}_i(\bigvee \Phi)$. In addition, because for all $\phi \in \Phi$, there is $t \in R_i(s)$ s.t. $M, t \models \phi$, so we have for all $\phi \in \Phi$, $M, s \models \hat{\mathbf{K}}_i \phi$. Hence, $M, s \models \bigwedge \hat{\mathbf{K}}_i \Phi$. $\square$

Proposition 2. *Suppose that ϕ is $\bot$ when Ψ is empty. Then*

$$\mathbf{K}_i\phi \wedge (\bigwedge \hat{\mathbf{K}}_i\Psi) \equiv \nabla_i cov(\phi, \Psi),\ \textit{where}$$

$$cov(\phi,\Psi) \doteq \begin{cases} \emptyset, & \textit{if } \phi = \bot; \\ \{\phi\} \cup \{\phi \wedge \psi \mid \psi \in \Psi\}, & \textit{otherwise.} \end{cases}$$

Proof. Firstly, suppose that Ψ is empty. Then, $\mathrm{cov}(\phi, \Psi) = \emptyset$ and $\phi = \bot$. So $\mathbf{K}_i\phi \wedge (\bigwedge \hat{\mathbf{K}}_i\Psi) \equiv \mathbf{K}_i\bot \wedge (\bigwedge \hat{\mathbf{K}}_i\emptyset) \equiv \mathbf{K}_i(\bigvee \emptyset) \wedge (\bigwedge \hat{\mathbf{K}}_i\emptyset) \equiv \nabla_i\emptyset \equiv \nabla_i\mathrm{cov}(\phi, \Psi)$.

Now, consider the case that Ψ is not empty.

1. It is easy to get that $\bigvee \mathrm{cov}(\phi, \Psi) \equiv \phi \vee \bigvee_{\psi\in\Psi}(\phi \wedge \psi)$. Since $\phi \wedge \psi \models \phi$ for each $\psi \in \Psi$, we have $\phi \vee \bigvee_{\psi\in\Psi}(\phi \wedge \psi) \equiv \phi$. Hence, $\bigvee \mathrm{cov}(\phi, \Psi) \equiv \phi$.
2. For each $\psi \in \Psi$, we have $\mathbf{K}_i\phi \wedge \hat{\mathbf{K}}_i\psi \equiv \mathbf{K}_i\phi \wedge \hat{\mathbf{K}}_i\phi \wedge \hat{\mathbf{K}}_i(\phi \wedge \psi)$. Thus $\mathbf{K}_i\phi \wedge (\bigwedge \hat{\mathbf{K}}_i\Psi) \equiv \mathbf{K}_i\phi \wedge [\bigwedge \hat{\mathbf{K}}_i(\{\phi\} \cup \{\phi \wedge \psi \mid \psi \in \Psi\})] \equiv \mathbf{K}_i(\bigvee \mathrm{cov}(\phi, \Psi)) \wedge [\bigwedge \hat{\mathbf{K}}_i(\mathrm{cov}(\phi, \Psi))] \equiv \nabla_i\mathrm{cov}(\phi, \Psi)$. □

Proposition 3. *Consider the context of a modal system* L. *Let $\delta \in E^P_k(\mathsf{L})$ where $k \in \mathbb{N}$ and $P \subseteq \mathcal{P}$ be finite. Let $\phi \in \mathcal{L}^{\mathbf{K}}_n$ s.t. $dep(\phi) \leq k$ and $\mathcal{P}(\phi) \subseteq P$. Then either $\delta \models \phi$ or $\delta \models \neg\phi$.*

Proof. We prove by induction on ϕ.

Case $\phi = p$: Obviously, $w(\delta)$ is a minterm of P. Since $p \in P$, either $w(\delta) \models p$ or $w(\delta) \models \neg p$.

Case $\phi = \neg\phi'$: By the induction hypothesis, if $\delta \models \phi'$, then $\delta \models \neg\phi$; otherwise, $\delta \models \phi$.

Case $\phi = \phi_1 \wedge \phi_2$: If $\delta \models \phi_1$ and $\delta \models \phi_2$, then $\delta \models \phi_1 \wedge \phi_2$. Otherwise, we have $\delta \models \neg\phi_1$ or $\delta \models \neg\phi_2$. We get $\delta \models \neg\phi_1 \vee \neg\phi_2$, *i.e.*, $\delta \models \neg(\phi_1 \wedge \phi_2)$.

Case $\phi = \mathbf{K}_i\phi'$: Let (M, s) be a model of δ. By the induction hypothesis, for every $\delta_i \in R_i(\delta)$, we have either $\delta_i \models \phi'$ or $\delta_i \models \neg\phi'$. We have two situations. Firstly, we assume that there exists $\delta_i \in R_i(\delta)$ s.t. $\delta_i \models \neg\phi'$. By the semantics of the cover modality, there exists $t \in R_i(s)$ s.t. $M, t \models \delta_i$. So $M, t \models \neg\phi'$ and $M, s \models \hat{\mathbf{K}}_i(\neg\phi')$, *i.e.*, $M, s \models \neg\mathbf{K}_i\phi'$. Secondly, we assume that for every $\delta_i \in R_i(\delta)$, $\delta_i \models \phi'$. By the semantics of the cover modality, for every $t \in R_i(s)$, there exists $\delta_i \in R_i(\delta)$ s.t. $M, t \models \delta_i$, and hence $M, t \models \phi'$. So $M, s \models \mathbf{K}_i\phi'$. □

Proposition 4. *Consider the context of a modal system* L. *Let $\delta \in E^P_k(\mathsf{L})$ where $k \in \mathbb{N}$ and $P \subseteq \mathcal{P}$ is finite. Let $\phi \in \mathcal{L}^{\mathbf{K}}_n$ s.t. $dep(\phi) \leq k$ and $\mathcal{P}(\phi) \subseteq P$. Then we can check if $\delta \models \phi$ recursively as follows:*

- *$\delta \models p$ iff p appears positively in $w(\delta)$;*
- *$\delta \models \neg\phi$ iff $\delta \not\models \phi$;*
- *$\delta \models \phi \wedge \psi$ iff $\delta \models \phi$ and $\delta \models \psi$;*
- *$\delta \models \phi \vee \psi$ iff $\delta \models \phi$ or $\delta \models \psi$;*
- *$\delta \models \mathbf{K}_i\phi$ iff for all $\delta' \in R_i(\delta)$, $\delta' \models \phi$.*

Proof. We only prove the cases of negation, disjunction and knowledge. The other two cases are trivial.

- Negation: By Proposition 3, $\delta \models \neg\phi$ iff $\delta \not\models \phi$.
- Disjunction: ($\Leftarrow$) Trivial. ($\Rightarrow$) Assume that $\delta \not\models \phi$ and $\delta \not\models \psi$. By Proposition 3, $\delta \models \neg\phi$ and $\delta \models \neg\psi$. Thus $\delta \models \neg\phi \wedge \neg\psi$, *i.e.*, $\delta \models \neg(\phi \vee \psi)$, contradicting $\delta \models \phi \vee \psi$. So $\delta \models \phi$ or $\delta \models \psi$;
- Knowledge: ($\Leftarrow$) Suppose for all $\delta' \in R_i(\delta)$, $\delta' \models \phi$. Let $M, s \models \delta$. We prove that $M, s \models K_i\phi$. Let $t \in R_i(s)$. Then there is $\delta' \in R_i(\delta)$ s.t. $M, t \models \delta'$. Since $\delta' \models \phi$, $M, t \models \phi$. So $M, s \models K_i\phi$. Thus $\delta \models K_i\phi$.
  ($\Rightarrow$) Suppose $\delta \models \mathbf{K}_i\phi$. Let $\delta' \in R_i(\delta)$. We prove $\delta' \models \phi$. Let $M, s \models \delta'$. Since δ is satisfiable, there exists a model (M_0, s_0) of δ. Without loss of generality, we assume that M_0 is disjoint from M. Since $\delta' \in R_i(\delta)$, there exists $t \in R_i(s_0)$ s.t. $M, t \models \delta'$. Now we let M'_0 be the same as M_0 except that we replace the i-edge from s_0 to t by one from s_0 to s. Then we have $M'_0, s_0 \models \delta$. Since $\delta \models \mathbf{K}_i\phi$, $M'_0, s_0 \models \mathbf{K}_i\phi$. Hence $M, s \models \phi$. Thus $\delta' \models \phi$. □

Proposition 6. *Consider the context of a modal system* L. *Let $\phi \in \mathcal{L}^{\mathbf{K}}_n$, $k \geq dep(\phi)$ and $P = \mathcal{P}(\phi)$. Then there exists a unique set $\Phi \subseteq E^P_k(\mathsf{L})$ s.t. $\phi \equiv \bigvee \Phi$.*

Proof. Existence: Let $\Phi = \{\delta \mid \delta \models \phi$ and $\delta \in E^P_k(\mathsf{L})\}$. Since $\delta \models \phi$ for $\delta \in \Phi$, we have $\bigvee \Phi \models \phi$. And for $\delta' \in E^P_k(\mathsf{L}) \setminus \Phi$, $\delta' \models \neg\phi$ (Proposition 3). So $\phi \equiv \bigvee_{\delta\in E^P_k(\mathsf{L})}(\phi \wedge \delta) \equiv \bigvee_{\delta\in\Phi}(\phi \wedge \delta) \equiv \bigvee \Phi$.

Uniqueness: Suppose that Φ and Φ' are two different canonical formulas of $E^P_k(\mathsf{L})$ satisfying the requirement. There exists a $\delta \in (\Phi \setminus \Phi') \cup (\Phi' \setminus \Phi)$. Without loss of generality, we assume $\delta \in \Phi \setminus \Phi'$. Since δ is satisfiable, there exists a model (M, s) of δ. Because $\phi \equiv \bigvee \Phi$, so $M, s \models \phi$. On the other hand, since $\delta \notin \Phi'$, we get that there does not exist a $\delta' \in \Phi'$ s.t. $M, s \models \delta'$ (Proposition 5), *i.e.*, $M, s \not\models \bigvee \Phi'$. Because $\phi \equiv \bigvee \Phi'$, so $M, s \not\models \phi$. Contradiction! □

Proposition 7. *Let δ be a canonical formula and $l < dep(\delta)$. Then for all $i \in \mathcal{A}$, we have $R_i(\delta^{\downarrow l}) = (R_i(\delta))^{\downarrow l}$.*

Proof. Firstly, we prove the case where $l = 1$. Since $dep(\delta) > l = 1$, we get $\delta^{\downarrow} = w(\delta) \wedge \bigwedge_{i\in\mathcal{A}} \nabla_i(R_i(\delta))^{\downarrow}$. Obviously, $R_i(\delta^{\downarrow}) = (R_i(\delta))^{\downarrow}$.

Now, we prove that $(R_i(\delta^{\downarrow l-k}))^{\downarrow k} = (R_i(\delta^{\downarrow l-k-1}))^{\downarrow k+1}$ for $0 \le k < l$. By Definition 8, we can get that $(R_i(\delta^{\downarrow l-k}))^{\downarrow k} = [R_i((\delta^{\downarrow l-k-1})^{\downarrow})]^{\downarrow k}$. Since $dep(\delta) > l$, $dep(\delta^{\downarrow l-k-1}) > l-(l-k-1) = k+1 > 1$. So $R_i((\delta^{\downarrow l-k-1})^{\downarrow}) = (R_i(\delta^{\downarrow l-k-1}))^{\downarrow}$. We get that $(R_i(\delta^{\downarrow l-k}))^{\downarrow k} = [R_i((\delta^{\downarrow l-k-1})^{\downarrow})]^{\downarrow k} = [(R_i(\delta^{\downarrow l-k-1}))^{\downarrow}]^{\downarrow k} = (R_i(\delta^{\downarrow l-k-1}))^{\downarrow k+1}$.

Hence, $R_i(\delta^{\downarrow l}) = (R_i(\delta^{\downarrow l-1}))^{\downarrow 1} = \cdots = (R_i(\delta^{\downarrow 1}))^{\downarrow l-1} = (R_i(\delta))^{\downarrow l}$. □

Proposition 8. *Let $\phi \in \mathcal{L}^{\mathbf{K}}_{\mathbf{C}}$ wherein p does not appear. Then, $(M,s)\underline{\leftrightarrow}_p(M',s')$ implies that $M,s \models \phi$ iff $M',s' \models \phi$.*

Proof. We prove by induction on ϕ. Let $M = \langle S, R, V\rangle$, $M' = \langle S', R', V'\rangle$, and ρ be the p-bisimulation between (M,s) and (M',s'). Here we only prove the only-if direction. The other direction can be similarly proved.

Case $\phi = q$ where $q \neq p$: Since $(M,s)\underline{\leftrightarrow}_p(M',s')$, we get $V(s) \sim_p V'(s')$. It is obvious that $V(s) \models q$ iff $V'(s') \models q$.

Case $\phi = \neg\phi'$: By the assumption, we get that $M,s \models \neg\phi'$, *i.e.*, $M,s \not\models \phi'$. Obviously, ϕ' does not contain any occurrence of p. By the induction hypothesis, we get that $M',s' \not\models \phi'$, *i.e.*, $M',s' \models \neg\phi'$.

Case $\phi = \phi_1 \wedge \phi_2$: By the assumption, we get that $M,s \models \phi_1$ and $M,s \models \phi_2$. Obviously, neither ϕ_1 nor ϕ_2 contains any occurrence of p. By the induction hypothesis, we get that $M',s' \models \phi_1$ and $M',s' \models \phi_2$. Hence, $M',s' \models \phi_1 \wedge \phi_2$.

Case $\phi = \mathbf{K}_i\phi'$: By the assumption, for every $t \in R_i(s)$, we have $M,t \models \phi'$. Obviously, ϕ' does not contain any occurrence of p. Let $t' \in R'_i(s')$. By the back condition of Definition 11, there exists $t \in R_i(s)$ s.t. $t\rho t'$. So $(M,t)\underline{\leftrightarrow}_p(M',t')$. This, together with the induction hypothesis, implies that $M',t' \models \phi'$. Hence, $M,s \models \mathbf{K}_i\phi'$.

Case $\phi = \mathbf{C}\phi'$: By the assumption, for every $t \in R_{\mathcal{A}}(s)$, we have $M,t \models \phi'$. Obviously, ϕ' does not contain any occurrence of p. Let $t' \in R'_{\mathcal{A}}(s')$. By repeatedly applying the back condition of Definition 11, we find a world $t \in R_{\mathcal{A}}(s)$ s.t. $t\rho t'$. The rest of proof is the same as the above case. □

Proposition 9. *Consider the context of a modal system* L. *If kforget$(\phi, p) \equiv \psi$, then the following hold:*

1. *$\phi \models \psi$;*
2. *for any formula η wherein p does not appear, $\phi \models \eta$ iff $\psi \models \eta$.*

Proof.

1. Let (M,s) be a model of ϕ. Obviously, $(M,s)\underline{\leftrightarrow}_p(M,s)$. By the forth condition of Definition 12, we have $M,s \models \psi$.
2. ($\Rightarrow$) Let (M',s') be a model of ψ. By the back condition of Definition 12, there exists a model (M,s) s.t. $(M,s)\underline{\leftrightarrow}_p(M',s')$ and $M,s \models \phi$. Since $\phi \models \eta$, we get that $M,s \models \eta$. Because $(M,s)\underline{\leftrightarrow}_p(M',s')$ and η is a formula wherein p does not appear, we have $M',s' \models \eta$.
   ($\Leftarrow$) Let (M,s) be a model of ϕ. By the first item, we get that $M,s \models \psi$. Since $\psi \models \eta$, we have $M,s \models \eta$. □

Proposition 10. *Consider the context of a modal system* L. *The following hold:*

1. *If $\phi \in \mathcal{L}^{pl}$ and pforget$(\phi, p) \equiv \psi$, then kforget$(\phi, p) \equiv \psi$;*
2. *kforget$(\phi_1 \vee \phi_2, p) \equiv$ kforget$(\phi_1, p) \vee$ kforget(ϕ_2, p).*

Proof.

1. **Forth:** Let (M,s) be a model of ϕ, and (M',s') be a model s.t. $(M,s)\underline{\leftrightarrow}_p(M',s')$. By Definition 11, we get that $V(s) \sim_p V'(s')$. Since $\phi \in \mathcal{L}^{pl}$, by Definition 10, $V'(s') \models \psi$. Hence, $M',s' \models \psi$.
   Back: Let (M',s') be a model of ψ. We get the valuation $V'(s')$. By Definition 10, there exists a valuation V^* s.t. $V^* \sim_p V'(s)$ and $V \models \phi$. Now, we let the structure of M be a copy of that of M', and let $V(s) = V^*$. Then we have $M,s \models \phi$ and $(M,s)\underline{\leftrightarrow}_p(M',s')$.
2. ($\Rightarrow$) Let (M',s') be a model of kforget$(\phi_1 \vee \phi_2, p)$. By the back condition of Definition 12, there exists a model (M,s) s.t. $(M,s)\underline{\leftrightarrow}_p(M',s')$ and $M,s \models \phi_1 \vee \phi_2$. Without loss of generality, we assume that $M,s \models \phi_1$. By the forth condition of Definition 12, we have $M',s' \models$ kforget(ϕ_1, p). Hence, $M',s' \models$ kforget$(\phi_1, p) \vee$ kforget(ϕ_2, p).
   ($\Leftarrow$) Let (M',s') be a model of kforget$(\phi_1, p) \vee$ kforget(ϕ_2, p). Without loss of generality, we assume that $M',s' \models$ kforget(ϕ_1, p). By the back condition of Definition 12, there exists a model (M,s) s.t. $(M,s)\underline{\leftrightarrow}_p(M',s')$ and $M,s \models \phi_1$. Moreover, $M,s \models \phi_1 \vee \phi_2$. By the forth condition of Definition 12, we have $M',s' \models$ kforget$(\phi_1 \vee \phi_2, p)$. □

Proposition 11. *If a modal system* L *is closed under forgetting, then* L *has uniform interpolation.*

Proof. Consider the context of a modal system L. Let ϕ be a formula and Q a set of variables. Firstly, we define the result of forgetting a set of variables as follows:

$$\text{kforget}(\phi, Q) = \begin{cases} \phi, & \text{if } Q = \emptyset; \\ \text{kforget}(\phi, p), & \text{if } Q = \{p\}; \\ \text{kforget}(\text{kforget}(\phi, Q \setminus \{p\}), p), & \text{otherwise.} \end{cases}$$

Let $P \subseteq \mathcal{P}(\phi)$ and define $\psi \equiv \text{kforget}(\phi, \mathcal{P}(\phi) \setminus P)$ to be the uniform interpolant of ϕ w.r.t. P. Obviously, ψ contains only atoms of P. Now, we prove that for any η s.t. $\mathcal{P}(\eta) \subseteq P$, we have $\phi \models \eta$ iff $\psi \models \eta$. We prove by induction on P.

Base case: Suppose that $p \in \mathcal{P}(\phi)$ and $P = \mathcal{P}(\phi) \setminus \{p\}$. By Proposition 9, we get that $\phi \models \eta$ iff $\psi \models \eta$.

Induction step: Suppose that P is non-empty. Let $\gamma = \text{kforget}(\phi, \mathcal{P}(\phi) \setminus P)$. Assume that γ is a uniform interpolant of ϕ w.r.t. P. Let $p \in P$ and $\psi = \text{kforget}(\phi, (\mathcal{P}(\phi) \setminus P) \cup \{p\})$. We want to prove that ψ is a uniform interpolant of ϕ w.r.t. $P \setminus \{p\}$.

($\Rightarrow$) Let (M', s') be a model of ψ. By the back condition of Definition 12, we get a model (M, s) of γ s.t. $(M, s) \underline{\leftrightarrow}_p (M', s')$. Since $\phi \models \eta$, $\mathcal{P}(\eta) \subseteq P \setminus \{p\} \subset P$ and the induction hypothesis, we have $\gamma \models \eta$. So $M, s \models \eta$. This, together with Proposition 8, implies that $M', s' \models \eta$.

($\Leftarrow$) By the induction hypothesis and Proposition 9, we have $\gamma \models \psi$. Hence, $\gamma \models \psi \models \eta$. $\square$

Proposition 12. *Let δ be a canonical formula and p an atom. Then, $\delta \models \delta^p$.*

Proof. We prove by induction on $dep(\delta)$. Let (M, s) be a model of δ.

Base case: Suppose that $\delta = \bigwedge_{q \in \tilde{P}} q \wedge \bigwedge_{q \in P \setminus \tilde{P}} \neg q$ where $\tilde{P} \subseteq P$. Then, $\delta^p = \bigwedge_{q \in \tilde{P} \setminus \{p\}} q \wedge \bigwedge_{q \in P \setminus (\tilde{P} \cup \{p\})} \neg q$. Since $M, s \models \delta$, for each $q \in \tilde{P} \setminus \{p\}$, $q \in V(s)$. Similarly, for each $q \in P \setminus (\tilde{P} \cup \{p\})$, $q \notin V(s)$. So $M, s \models \delta^p$.

Induction step: Now, we show that $M, s \models w(\delta^p) \wedge \bigwedge_{i \in \mathcal{A}} \nabla_i R_i(\delta^p)$. Firstly, by the base case, we have $w(\delta) \models w(\delta^p)$. Now, we prove that for any $i \in \mathcal{A}$, $M, s \models \nabla_i R_i(\delta^p)$. We only prove the forth condition that for all $t \in R_i(s)$, there is $\delta_i^p \in R_i(\delta^p)$ s.t. $M, t \models \delta_i$. The back condition can be similarly proved. By the semantics of ∇_i modality, there is $\delta_i \in R_i(\delta)$ s.t. $M, t \models \delta_i$. By the induction hypothesis, we get that $\delta_i \models \delta_i^p$, and hence $M, t \models \delta_i^p$. Obviously, $\delta_i^p \in R_i(\delta^p)$. $\square$

Proposition 13. *Let δ be a $\mathsf{T_n}$ satisfiable canonical formula where $dep(\delta) \geq 1$. Let $l \in \mathbb{N}$ s.t. $1 \leq l \leq dep(\delta)$. Then, for all $i \in \mathcal{A}$, we have $\delta^{\downarrow l} \in R_i(\delta^{\downarrow l-1})$.*

Proof. Let $k = dep(\delta)$, $P = \mathcal{P}(\delta)$ and $M, s \models \delta$. Obviously, M, s satisfies both $\delta^{\downarrow l}$ and $\delta^{\downarrow l-1}$. Since M is reflexive, $s \in R_i(s)$. By the semantics of the ∇_i modality, there exists $\delta_i \in R_i(\delta^{\downarrow l-1})$ s.t. $M, s \models \delta_i$. Since $\delta \in E_k^P$, it is easily verified that $\delta^{\downarrow l} \in E_{k-l}^P$. Similarly, $\delta^{\downarrow l-1} \in E_{k-l+1}^P$, and hence $\delta_i \in E_{k-l}^P$. So both $\delta^{\downarrow l}$ and δ_i are the depth $k - l$ canonical formula of (M, s) (cf. Proposition 5). Hence $\delta^{\downarrow l} = \delta_i \in R_i(\delta^{\downarrow l-1})$. $\square$

Proposition 5. *Let (M, s) be a Kripke model and $k \in \mathbb{N}$. Let $P \subseteq \mathcal{P}$ be finite. Then, there exists a unique $\delta \in E_k^P$ s.t. $M, s \models \delta$.*

Since Proposition 14 is a generalization of Proposition 5, below we prove them together.

Proposition 14. *Let (M, T) be a multi-pointed model and $k \in \mathbb{N}$. Let $P \subseteq \mathcal{P}$ be finite. Then, there exists a unique set $\Phi \subseteq E_k^P$ s.t. (M, T) is Φ-complete.*

Proof. Here we show Propositions 5 and 14 together.

Firstly, we prove that if Proposition 5 holds for the case E_k^P, then Proposition 14 also holds for the same case.

Existence: Let $\Phi = \{\delta \mid \delta \in E_k^P$ and $M, t \models \delta$ and $t \in T\}$. It is easy to check that (M, T) is Φ-complete.

Uniqueness: Suppose that Φ and Φ' are two different sets of E_k^P. The multi-pointed model (M, T) is both Φ-complete and Φ'-complete. Since $\Phi \neq \Phi'$, there exists $\delta \in (\Phi \setminus \Phi') \cup (\Phi' \setminus \Phi)$. Without loss of generality, we suppose that $\delta \in (\Phi \setminus \Phi')$. Since (M, T) is Φ-complete, there exists $t \in T$ s.t. $M, t \models \delta$. Similarly, there exists $\delta' \in \Phi'$ s.t. $M, t \models \delta'$. Obviously, δ and δ' are k depth canonical formulas of (M, t). By Proposition 5, we get $\delta = \delta'$. So $\delta \in \Phi'$. Contradiction!

Secondly, we prove that Proposition 5 holds.

Base case: Suppose that $M = \langle S, R, V \rangle$, we define $\delta = \bigwedge_{p \in P \cap V(s)} p \wedge \bigwedge_{p \in P \setminus V(s)} \neg p$. Obviously, δ is the unique canonical formula of E_0^P s.t. $M, s \models \delta$.

Induction step: Assume that Propositions 5 and 14 both hold for the case E_k^P. **Existence:** By the base case, there exists a unique $\delta_0 \in E_0^P$ s.t. $M, s \models \delta_0$. For each $i \in \mathcal{A}$, there exists a unique set $\Phi_i \subseteq E_k^P$ s.t. $(M, R_i(s))$ is Φ_i-complete. Let $\delta = \delta_0 \wedge \bigwedge_{i \in \mathcal{A}} \nabla_i \Phi_i$. Obviously, $M, s \models \delta$. **Uniqueness:** Suppose that δ and δ' are two different canonical formulas of E_{k+1}^P. Let $\delta' = \delta_0' \wedge \bigwedge_{i \in \mathcal{A}} \nabla_i \Phi_i'$. So $\delta_0 \neq \delta_0'$, or there exists an agent i s.t. $\Phi_i \neq \Phi_i'$. Since both δ_0 and δ_0' are satisfied by (M, s), by the uniqueness of the base case of Proposition 5, we get that $\delta_0 = \delta_0'$. We consider the second situation that $\Phi_i \neq \Phi_i'$ for some agent i. Since $M, s \models \nabla_i \Phi_i$ and $M, s \models \nabla_i \Phi_i'$, we get that $(M, R_i(s))$ is Φ_i-complete and Φ_i'-complete. Because Φ_i and Φ_i' are depth k canonical formula sets, so they are equal. Hence, both situations are impossible. Contradiction! $\square$

Proposition 15. *Let $\Phi \subseteq \mathcal{L}_{\mathbf{C}}^{\mathbf{K}}$ such that Φ is finite and every $\phi \in \Phi$ does not contain any occurrence of p. Then, $(M,T)\underline{\leftrightarrow}_p(M',T')$ implies that (M,T) is Φ-complete iff (M',T') is Φ-complete.*

Proof. Here, we only prove the only-if direction of the first condition of Φ-completeness (Definition 19). The other proofs are similar. Now, we prove that for every $t' \in T'$, there exists $\phi \in \Phi$ s.t. $M',t' \models \phi$. Let ρ be the p-bisimulation between the worlds of M and those of M'. There exists $t \in T$ s.t. $\rho(t,t')$. Since (M,T) is Φ-complete, there exists $\phi \in \Phi$ s.t. $M,t \models \phi$. Obviously, $(M,t)\underline{\leftrightarrow}_p(M',t')$. Hence, $M',t' \models \phi$. □

Proposition 16. *Let δ be a $\mathsf{K45_n}$ satisfiable canonical formula where $dep(\delta) \geq 2$. Let $l \in \mathbb{N}$ s.t. $1 \leq l < dep(\delta)$. Then, for all $i \in \mathcal{A}$ and $\delta_i \in R_i(\delta)$, $(R_i(\delta))^{\downarrow l} = R_i(\delta_i^{\downarrow l-1})$.*

Proof. Let $k = dep(\delta)$ and $M,s \models \delta$. Then there exists $t \in R_i(s)$ s.t. $M,t \models \delta_i$. Since M is transitive and Euclidean, $R_i(s) = R_i(t)$. Obviously, $(M,R_i(s))$ is $R_i(\delta)$-complete and $(M,R_i(t))$ is $R_i(\delta_i)$-complete. Then, $(M,R_i(s))$ is $(R_i(\delta))^{\downarrow l}$-complete and $(M,R_i(t))$ is $R_i(\delta_i^{\downarrow l-1})$-complete. So $(M,R_i(s))$ is also $R_i(\delta_i)$-complete. So both $(R_i(\delta))^{\downarrow l}$ and $R_i(\delta_i^{\downarrow l-1})$ are the depth $k-l-1$ canonical formula set of $(M,R_i(s))$ (cf. Proposition 14). Hence $(R_i(\delta))^{\downarrow l} = R_i(\delta_i^{\downarrow l-1})$. □

Lemma 1 *(The $\mathsf{K45_n}$ lemma). Let δ be a $\mathsf{K45_n}$ satisfiable canonical formula where $dep(\delta) \geq 1$. Let (M',s') be a $\mathsf{K45_n}$ model of δ^p. Then for all $i \in \mathcal{A}$, there exists a multi-pointed $\mathsf{K45_n}$ model (M,T) that is i-equivalent, $R_i(\delta)$-complete and p-bisimilar to $(M',R_i'(s'))$.*

Proof. From now on, we let T' be $R_i'(s')$. Here, we show that (M,T), constructed in the main body of this paper, is i-equivalent, transitive, Euclidean, p-bisimilar to (M',T'), and $R_i(\delta)$-complete.

Firstly, we only let $t_1R_it_2$ where $t_1,t_2 \in T$. So (M,T) is i-equivalent.

Secondly, we show that M is transitive. Suppose that $u,v,w \in S$, uR_jv and vR_jw. We prove that uR_jw in the following. Recall that for every $t \in T$ and every agent $k \neq i$, we construct a submodel $(M_{t,k},T_{t,k})$ where $M_{t,k} = \langle S_{t,k},R_{t,k},V_{t,k}\rangle$. For the base case, if $u \in S_{t,k}$, then we use u' to denote the original world u of S'. Both v' and w' are similar.

1. $u,v,w \in T$: Since we only add i-edges between all worlds of T, we get that $j = i$. Also (M,T) is i-equivalent, so uR_iw.
2. $u,v \in T$ and $w \in S_{t,k}$ where $t \in T$ and $k \in \mathcal{A}$: Because we only add j-edges between v and all worlds of $T_{v,j}$ where $j \neq i$, so $t = v$ and $j = k \neq i$. On the other hand, similarly to Case 1, we get that $j = i$ since uR_jv when $j = i$. Hence this case is impossible, *i.e.*, uR_jv and vR_jw cannot hold simultaneously.
3. $u \in S$, $v \in S_{t,k}$ and $w \in T$ where $t \in T$ and $k \in \mathcal{A}$: This case is impossible since we do not add any edge from the worlds of $S_{t,k}$ to those of T, *i.e.*, the assumption that vR_jw does not hold.
4. $u \in T$, $v \in S_{t,k}$ and $w \in S_{x,l}$ where $t,x \in T$ and $k,l \in \mathcal{A}$: Similarly to Case 2, we get $t = u$ and $j = k \neq i$. We get that $x = t = u$ and $l = k = j$ since we do not add any edge from the worlds of $S_{t,k}$ to those of $S_{x,l}$ where $t \neq x$ or $k \neq l$. Since uR_jv, there exists a submodel $(M_{u,j},T_{u,j})$ s.t. $v \in T_{u,j}$. In the following, we prove by induction on $dep(\delta)$. **Base case:** By the construction, $(M_{u,j},T_{u,j})$ is a copy of $(M',R_j'(u'))$. So we get that $u'R_j'v'$ and $v'R_j'w'$. These, together with the transitivity of M', imply that $u'R_j'w'$. By the construction, we get that $w \in T_{u,j}$, and hence uR_jw. **Induction step:** By the induction hypothesis, $(M_{u,j},T_{u,j})$ is j-equivalent. Since vR_jw, $w \in T_{u,j}$. By the construction, we get that uR_jw.
5. $u \in S_{t,k}$, $v \in T$ and $w \in S$ where $t \in T$ and $k \in \mathcal{A}$: Similarly to case 3, this case is impossible since uR_jv does not hold.
6. $u \in S_{t,k}$, $v \in S_{x,l}$ and $w \in S_{y,m}$ where $t,x,y \in T$ and $k,l,m \in \mathcal{A}$: Similarly to Case 2, we get that $t = x = y$ and $k = l = m$. There exists a submodel $(M_{t,j},T_{t,j})$ where $M_{t,j} = \langle S_{t,j},R_{t,j},V_{t,j}\rangle$ s.t. $u,v,w \in S_{t,j}$. In the following, we prove vR_jw by induction on $dep(\delta)$. **Base case:** Similarly to Case 4, we get that $u'R_j'v'$ and $v'R_j'w'$ since $(M_{t,j},T_{t,j})$ is a copy of $(M',R_j'(t'))$. Because M' is transitive, so $u'R_j'w'$. Hence, we get that uR_jw. **Induction step:** By the hypothesis induction, $M_{t,k}$ is transitive. Since $u,v,w \in S_{t,k}$, we have uR_jw if uR_jv and vR_jw both hold.

Thirdly, we show that M is Euclidean. Suppose that $u,v,w \in S$, uR_jv and uR_jw. We prove that vR_jw in the following.

1. $u,v,w \in T$: Similarly to Case 1 of the transitivity property of M, we get $j = i$ and vR_jw.
2. $u,v \in T$ and $w \in S_{t,k}$ where $t \in T$ and $k \in \mathcal{A}$: Similarly to Case 2 of the transitivity property of M, we get that uR_jv and uR_jw are inconsistent. This case is impossible.
3. $u,w \in T$ and $v \in S_{t,k}$ where $t \in T$ and $k \in \mathcal{A}$: Similarly to Case 2, this case is impossible.
4. $u \in T$, $v \in S_{t,k}$ and $w \in S_{x,l}$ where $t,x \in T$ and $k,l \in \mathcal{A}$: Similarly to Case 4 of the transitivity property of M, there exists a submodel $(M_{u,j},T_{u,j})$ where $M_{u,j} = \langle S_{u,j},R_{u,j},V_{u,j}\rangle$ s.t. $v,w \in T_{u,j}$. By the construction, we have vR_jw.
5. $u \in S_{t,k}$, $v \in T$ and $w \in S$ where $t \in T$ and $k \in \mathcal{A}$: Similarly to Case 3 of the transitivity property of M, this case is impossible since uR_jv does not hold.
6. $u \in S_{t,k}$, $v \in S$ and $w \in T$ where $t \in T$ and $k \in \mathcal{A}$: Similarly to the above case, this case is impossible.
7. $u \in S_{t,k}$, $v \in S_{x,l}$ and $w \in S_{y,m}$ where $t,x,y \in T$ and $k,l,m \in \mathcal{A}$: The proof is similar to that of Case 6 of the transitivity property of M except we use that M' is Euclidean.

Then, we show that ρ is a p-bisimulation between (M, T) and (M', T'). Obviously, for every $t \in T$, there exists $t' \in T'$ s.t. $t\rho t'$; and the other direction also holds.

Atom: Suppose that $u\rho u'$. If $u \in T$, then $u' \in T'$. By the construction, there exists a $\delta_i \in R_i(\delta)$ s.t. $u \models \delta_i$ and $u' \models \delta_i^p$. So $V(u) \sim_p V'(u')$. Otherwise, we have $u \notin T$. By the construction, there exists a p-bisimulation $\rho_{t,j}$ s.t. $(u, u') \in \rho_{t,j}$. So $V(u) \sim_p V'(u')$.

Forth: Suppose that $u, v \in S$ and $u' \in S'$ s.t. uR_jv and $(u, u') \in \rho$.

1. $u, v \in T$: Similarly to Case 1 of the proof of the transitivity property of M, we get that $j = i$ and uR_iv. According to Step 1 of the construction, we have $u' \in T'$, and there exists $v' \in S'$ s.t. $(v, v') \in \rho$ and $u'R_i'v'$.
2. $u \in T$ and $v \notin T$: Similarly to Case 1 of the transitivity property of M, we get that $j \neq i$. There exists a submodel $(M_{u,j}, T_{u,j})$ where $M_{u,j} = \langle S_{u,j}, R_{u,j}, V_{u,j}\rangle$ s.t. $v \in S_{u,j}$. According to Step 1 of the construction, we have $u' \in T'$. According to Step 2 of the construction, there exists a p-bisimulation $\rho_{u,j} : (M_{u,j}, T_{u,j}) \underline{\leftrightarrow}_p (M', R_j'(u'))$. By the forth condition of $\rho_{u,j}$, there exists $v' \in S'$ s.t. $(v, v') \in \rho_{u,j}$ and $u'R_j'v'$. Hence $(v, v') \in \rho$.
3. $u \notin T$ and $v \in T$: Similarly to Case 2 of the transitivity property of M, this case is impossible.
4. $u, v \notin T$: Similarly to Case 2, there exists a submodel $(M_{t,j}, T_{t,j})$ where $t \in T$ and $M_{t,j} = \langle S_{t,j}, R_{t,j}, V_{t,j}\rangle$ s.t. $u, v \in S_{t,j}$. Also, there exists a p-bisimulation $\rho_{t,j} : (M_{t,j}, T_{t,j}) \underline{\leftrightarrow}_p (M', R_j'(t'))$ where $t\rho t'$. Similarly to Case 4, there exists $v' \in S'$ s.t. $(v, v') \in \rho$ and $u'R_j'v'$.

Back: Suppose that $u \in S$ and $u', v' \in S'$ s.t. $u'R_j'v'$ and $(u, u') \in \rho$. Here, we only consider the case that $u', v' \in T'$. The other cases can be similarly proved.

Obviously, $u \in T$ or $u \in S_{t,k}$ where $t \in T$ and $k \in \mathcal{A}$. If $u \in T$ and $j = i$, then by the Step 1 of the construction, there exists a $v \in T$ s.t. $(v, v') \in \rho$ and uR_iv.

If $u \in T$ and $j \neq i$, then there exists a submodel $(M_{u,j}, T_{u,j})$ where $M_{u,j} = \langle S_{u,j}, R_{u,j}, V_{u,j}\rangle$ and a p-bisimulation $\rho_{u,j} : (M_{u,j}, T_{u,j}) \underline{\leftrightarrow}_p (M', R_j'(u'))$. By the back condition of $\rho_{u,j}$, there exists $v \in S_{u,j}$ s.t. $(v, v') \in \rho_{u,j}$ and uR_jv. Hence $(v, v') \in \rho$ and $v \in S$.

If $u \notin T$, then there exists a submodel $(M_{t,k}, T_{t,k})$ where $M_{t,k} = \langle S_{t,k}, R_{t,k}, V_{t,k}\rangle$ where $t \in T$ and $k \in \mathcal{A}$ s.t. $u \in S_{t,k}$. Also, there exists a p-bisimulation $\rho_{t,k} : (M_{t,k}, T_{t,k}) \underline{\leftrightarrow}_p (M', R_j'(t'))$ where $t\rho t'$. Similarly to the above situation, there exists $v \in S$ s.t. $(v, v') \in \rho$ and uR_jv.

Finally, we prove (M, T) is $R_i(\delta)$-complete by induction on $dep(\delta)$.

Base case: By Step 1 of the construction, it is easy to verify that (M, T) is $R_i(\delta)$-complete.

Induction step: Let $dep(\delta) = k$. We prove by induction on $k - l$ that for all $0 \le l < k$, (M, T) is $R_i(\delta)^{\downarrow l}$-complete. **Base case ($l = k - 1$):** The set $R_i(\delta)^{\downarrow l}$ is a set of propositional formulas. In Step 1, (M, T) is $R_i(\delta)^{\downarrow l}$-complete. **Induction step:** Assume that (M, T) is $(R_i(\delta))^{\downarrow l}$-complete. We prove that it is also $(R_i(\delta))^{\downarrow l-1}$-complete. Let $t \in T$. By the construction, there exists $t' \in T'$ and $\delta_i \in R_i(\delta)$ s.t. $t\rho t'$ and $M', t' \models \delta_i^p$. It suffices to show that $M, t \models \delta_i^{\downarrow l-1}$. Since $w(\delta_i^{\downarrow l-1}) = w(\delta_i)$, $M, t \models w(\delta_i^{\downarrow l-1})$. By the identical-children property (Proposition 16), $(R_i(\delta))^{\downarrow l} = R_i(\delta_i^{\downarrow l-1})$. By the induction hypothesis, (M, T) is $R_i(\delta_i^{\downarrow l-1})$-complete, so $M, t \models \nabla_i R_i(\delta_i^{\downarrow l-1})$. For $j \neq i$, because $(M_{t,j}, T_{t,j})$ is $R_j(\delta_i)$-complete, $R_j(t) = T_{t,j}$ and $M, t \models \nabla_j R_j(\delta_i)$, hence $M, t \models \nabla_j R_j(\delta_i)^{\downarrow l-1}$. By Proposition 7, $R_j(\delta_i)^{\downarrow l-1} = R_j(\delta_i^{\downarrow l-1})$, so $M, t \models \nabla_j R_j(\delta_i^{\downarrow l-1})$. $\square$

Proposition 17. *Let δ be an $\mathsf{S5_n}$ satisfiable canonical formula where $dep(\delta) \ge 2$. Let $l \in \mathbb{N}$ s.t. $2 \le l \le dep(\delta)$. Then, for all $i \in \mathcal{A}$ and $\delta_i \in R_i(\delta)$, $\delta^{\downarrow l} \in R_i(\delta_i^{\downarrow l-2})$.*

Proof. By the reflexive property (Proposition 13), $\delta^{\downarrow l} \in R(\delta^{\downarrow l-1})$. By Propositions 7 and 16, we get that $R(\delta^{\downarrow l-1}) = R_i(\delta_i^{\downarrow l-2})$. Hence, $\delta^{\downarrow l} \in R_i(\delta_i^{\downarrow l-2})$. $\square$

Lemma 2 *(The $\mathsf{S5_n}$ lemma). Let δ be an $\mathsf{S5_n}$ satisfiable canonical formula where $dep(\delta) \ge 1$. Let (M', s') be an $\mathsf{S5_n}$ model of δ^p. Then for all $i \in \mathcal{A}$, there exists a multi-pointed $\mathsf{S5_n}$ model (M, T) that is i-equivalent, $R_i(\delta)$-complete and p-bisimilar to $(M', R_i'(s'))$.*

Proof. To distinguish the models before and after adding reflexive and symmetric edges, we let (M^*, T^*) be the former where $M^* = \langle S^*, R^*, V^*\rangle$, and (M, T) be the latter where $M = \langle S, R, V\rangle$.

Here, we show that (M, T), constructed in the main body of this paper, satisfies the requirements. Obviously, (M, T) is still i-equivalent since Steps 2 and 3 do not add any i-edge between a world of T and another one which is not in T. Moreover, it is obvious that M satisfies reflexivity since all submodels are reflexive, and Step 2 adds reflexive edges for all worlds of T. Note that it is not necessary to verify that M is transitive and symmetric. This is because a model, which is reflexive and Euclidean, is also transitive and symmetric.

According to the proof of the $\mathsf{K45_n}$ lemma, we get that the model (M^*, T^*) is Euclidean, p-bisimilar to $(M', R_i'(s'))$, and $R_i(\delta)$-complete. But we add reflexive and symmetric edges for the model M^* and acquire the new model M. Hence, we need to verify that (M, T) satisfies the above three conditions.

Firstly, we prove that M is Euclidean. Suppose that $u, v, w \in S$, uR_jv and uR_jw. We prove that vR_jw in the following:

1. $u, v, w \in T$: If $j \neq i$, then $u = v = w$ and vR_jw since we only add reflexive edges for the worlds of T. Otherwise, we can get that vR_iw since (M, T) is i-equivalent.
2. $u, v \in T$ and $w \in S_{t,k}$ where $t \in T$ and $k \in \mathcal{A}$: By the construction, there exists only j-edges between u and all worlds of $T_{u,j}$ where $j \neq i$. So $t = u$ and $j = k \neq i$. Similarly to Case 1, if $j \neq i$, then $u = v$. By the assumption that uR_jw, we get vR_jw.
3. $u, w \in T$ and $v \in S_{t,k}$ where $t \in T$ and $k \in \mathcal{A}$: As the proof in Case 2, we can get that wR_jv, $t = u = w$ and $j = k$. Since we add symmetric edges for the model, we have vR_jw.
4. $v, w \in T$ and $u \in S_{t,k}$ where $t \in T$ and $k \in \mathcal{A}$: Since we only add a j-edge from u to t, we get that $t = v = w$ and $k = j$. Since we add reflexive edges for the world v and the agent j, we get that vR_jv, *i.e.*, vR_jw.
5. $u \in T$, $v \in S_{t,k}$ and $w \in S_{x,l}$ where $t, x \in T$ and $k, l \in \mathcal{A}$: The proof of this case is the same as that of Case 4 of the Euclidean property of Lemma 1.
6. $u \in S_{t,k}$, $v \in T$ and $w \in S_{x,l}$ where $t, x \in T$ and $k, l \in \mathcal{A}$: Similarly to Case 2, we get that $t = v$ and $j = k \neq i$ since uR_jv. By the construction, there exists a submodel $(M_{t,j}, T_{t,j})$ where $M_{t,j} = \langle S_{t,j}, R_{t,j}, V_{t,j}\rangle$ s.t. $u, w \in S_{t,j}$. In the following, we prove by induction on $dep(\delta)$. **Base case:** The model $(M_{t,j}, T_{t,j})$ is a copy of $(M', R'_j(t'))$. Let u' and w' be the original world of u and w in S'. Since uR_jv and uR_jw, we get that $u'R'_jv'$ and $u'R'_jw'$. Since M' is Euclidean, $v'R'_jw'$. So $w \in T_{v,j}$. By the construction, we get that vR_jw. **Induction step:** By the induction hypothesis, $(M_{t,j}, T_{t,j})$ is j-equivalent. Since uR_jw, so $w \in T_{t,j}$. By the construction, vR_jw.
7. $u \in S_{t,k}$, $v \in S_{x,l}$ and $w \in T$ where $t, x \in T$ and $k, l \in \mathcal{A}$: As in the proof of Case 6, we get that wR_jv and $v \in T_{w,j}$. In Step 2 of the construction illustrated in the $\mathsf{S5_n}$ lemma (Lemma 2), we add an j-edge from v to w. Hence, vR_jw.
8. $u \in S_{t,k}$, $v \in S_{x,l}$ and $w \in S_{y,m}$ where $t, x, y \in T$ and $k, l, m \in \mathcal{A}$: This proof is the same as that of Case 7 of the Euclidean property of Lemma 1.

Then, we prove that $(M, T)\underline{\leftrightarrow}_p(M', R'_i(s'))$. The proof of most cases of the forth condition and the atom and back conditions are the same as those of Lemma 1. Here, we only prove the following cases of the forth condition since we add reflexive and symmetric edges. Suppose that $u, v \in S$ and $u' \in S'$ s.t. uR_jv and $(u, u') \in \rho$.

1. $u, v \in T$ and $j \neq i$: Since we only add a j-edge from u to itself, we have $u = v$. Let $v' = u'$. Obviously, $(v, v') \in \rho$ and $u'R'_jv'$.
2. $u \notin T$ and $v \in T$: In Step 3, we only add an j-edge from u to v where $j \neq i$ and $u \in T_{v,j}$. Obviously, vR_ju holds. Since $v \in T$, by the construction, there exists a submodel $(M_{v,j}, T_{v,j})$ where $M_{v,j} = \langle S_{v,j}, R_{v,j}, V_{v,j}\rangle$ s.t. $u \in S_{v,j}$. In the following, we prove by induction on $dep(\delta)$ that the required v' exists.
   Base case: The model $(M_{v,j}, T_{v,j})$ is a copy of $(M', R'_i(v'))$ where $v\rho v'$. So u' is a copy of u and $v'R'_ju'$. Since M' is symmetric, we have $u'R'_jv'$.
   Induction step: In Step 1, we acquire the submodel $(M_{v,j}, T_{v,j})$ and the p-bisimulation $\rho_{v,j} : (M_{v,j}, T_{v,j})\underline{\leftrightarrow}_p(M', R'_i(v'))$ by the induction hypothesis of this lemma. By the forth condition of $\rho_{v,j}$, there exists $v' \in S'$ s.t. $u'R'_jv'$ and $(v, v') \in \rho_{v,j}$. Hence, $(v, v') \in \rho$.

Finally, we prove that (M, T) is $R_i(\delta)$-complete. Before we give a proof, we introduce some concepts as follows. Let $m = dep(\delta)$. For the model M, we define a path τ from a world s_0 to a world s_n of S to be $\langle s_0, i_0, \dots, i_{n-1}, s_n\rangle$ where for $0 \le j \le n$, $s_j \in S$ and for $0 \le j < n$, $i_j \in \mathcal{A}$ s.t. $s_jR_{i_j}s_{j+1}$. We use $len(\tau)$ to denote the length of τ and $\mathrm{e}(\tau)$ to denote the last world of τ. For example, let $\tau = \langle s_0, i_0, \dots, i_n, s_{n+1}\rangle$. Then, $len(\tau) = n + 1$ and $\mathrm{e}(\tau) = s_{n+1}$. Given two paths $\tau = \langle s_0, i_0, \dots i_{n-1}, s_n\rangle$ and $\tau' = \langle s_0, i_0, \dots i_{n-1}, s_n, i_n, s_{n+1}\rangle$, we say τ' is an i_n-extension of τ by s_{n+1}, denoted by $\tau + (i_n, s_{n+1})$.

Now, we recursively define a mapping from any path τ starting from a world $t \in T$ with at most length $m - 1$ to a canonical formula δ_τ and prove that $M^*, \mathrm{e}(\tau) \models \delta_\tau$ as follows:

Base case: Let $\tau = \langle t\rangle$ where $t \in T$, by the construction, there exists $t' \in T'$ and $\delta_i \in R_i(\delta)$ s.t. $t\rho t'$, $M', t' \models \delta_i^p$, and $M^*, t \models \delta_i$. We let $\delta_\tau = \delta_i$.

Inductive case: Let $\tau = \tau' + (j, u)$. By the induction hypothesis, we get that $M^*, \mathrm{e}(\tau') \models \delta_{\tau'}$. If $u \notin T$, then there exists $\delta_j \in R_j(\delta_{\tau'})$ s.t. $M^*, u \models \delta_j$. We let $\delta_\tau = \delta_j$. Otherwise, we have $u \in T$. We let $\delta_\tau = \delta_{\langle u\rangle}^{\downarrow len(\tau)}$.

By Proposition 5, we get the following fact: for any two paths τ and τ' s.t. $\mathrm{e}(\tau) = \mathrm{e}(\tau')$ and $len(\tau) \ge len(\tau')$, we have $\delta_\tau = \delta_{\tau'}^{\downarrow len(\tau)-len(\tau')}$.

Now, we prove that after adding edges, for every path τ, $M, \mathrm{e}(\tau) \models \delta_\tau$. Obviously, the above implies that (M, T) is $R_i(\delta)$-complete. We prove by induction on $m - l$ that for every $l < m$ and every path τ where $len(\tau) \le l$, we have $M, \mathrm{e}(\tau) \models \delta_\tau^{\downarrow l-len(\tau)}$.

Base case ($l = m - 1$): Let τ be a path where $len(\tau) \le m - 1$. Obviously, $dep(\delta_\tau^{\downarrow m-len(\tau)-1}) = 0$, *i.e.*, $\delta_\tau^{\downarrow m-len(\tau)-1}$ is propositional. Moreover, $\delta_\tau^{\downarrow m-len(\tau)-1} = w(\delta_\tau)$. By the definition of δ_τ, $M^*, \mathrm{e}(\tau) \models \delta_\tau$. So $M^*, \mathrm{e}(\tau) \models w(\delta_\tau)$. Because neither Step 2 nor 3 modifies the valuation on $\mathrm{e}(\tau)$, and $w(\delta_\tau)$ is propositional, so after Steps 2 and 3, $M, \mathrm{e}(\tau) \models w(\delta_\tau)$.

Induction step: Assume that for every path τ where $len(\tau) \le l$, we have $M, \mathrm{e}(\tau) \models \delta_\tau^{\downarrow l-len(\tau)}$. We want to prove that for every path τ where $len(\tau) \le l - 1$, we have $M, \mathrm{e}(\tau) \models \delta_\tau^{\downarrow l-len(\tau)-1}$. Let $u = \mathrm{e}(\tau)$. As in the proof of the base case, we get that

$M, \mathrm{e}(\tau) \models w(\delta_\tau)$ and hence $M, \mathrm{e}(\tau) \models w(\delta_\tau^{\downarrow l-len(\tau)-1})$. In the following, we show that $M, \mathrm{e}(\tau) \models \bigwedge_{j\in\mathcal{A}} \nabla_j R_j(\delta_\tau^{\downarrow l-len(\tau)-1})$. Here, we only prove that for every $v \in R_j(\mathrm{e}(\tau))$, there exists $\delta_j \in R_j(\delta_\tau^{\downarrow l-len(\tau)-1})$ s.t. $M, v \models \delta_j$. The back condition can be similarly proved. Let $\tau' = \tau + (j, v)$. By the induction hypothesis, we get that $M, v \models \delta_{\tau'}^{\downarrow l-len(\tau')}$ and hence $\delta_{\tau'}^{\downarrow l-len(\tau')}$ is the desired δ_j. It remains to prove that $\delta_{\tau'}^{\downarrow l-len(\tau')} \in R_j(\delta_\tau^{\downarrow l-len(\tau)-1})$. We analyze it by the following cases.

1. $u, v \in T$: Suppose that $u = v$. By the definition of δ_τ and $\delta_{\tau'}$, there exists $\delta_i \in R_i(\delta)$ s.t. $\delta_\tau = \delta_i^{\downarrow len(\tau)}$ and $\delta_{\tau'} = \delta_\tau^{\downarrow}$. By the reflexive property (Proposition 13), we have $\delta_{\tau'} \in R_j(\delta_\tau)$ and hence $\delta_{\tau'}^{\downarrow l-len(\tau')} = \delta_{\tau'}^{\downarrow l-len(\tau)-1} \in R_j(\delta_\tau)^{\downarrow l-len(\tau)-1}$.
   Suppose that $u \neq v$. Clearly, $len(\tau) \geq 1$ and hence $l \geq 1$. By the construction, we get that $j = i$. By the definition of δ_τ and $\delta_{\tau'}$, there exist $\delta_u, \delta_v \in R_i(\delta)$ s.t. $\delta_\tau = \delta_u^{\downarrow len(\tau)}$ and $\delta_{\tau'} = \delta_v^{\downarrow len(\tau')}$. By the identical-children property (Proposition 16), we have $\delta_{\tau'}^{\downarrow l-len(\tau')} = \delta_v^{\downarrow l} \in (R_i(\delta))^{\downarrow l} = R_i(\delta_u^{\downarrow l-1}) = R_i(\delta_\tau^{\downarrow l-len(\tau)-1})$.
2. $u \notin T$ and $v \in T$: It is obvious that $i \neq j$ and $v \in T_{u,j}$ since we only add j-edges from the worlds of $T_{u,j}$ to u where $j \neq i$. By the construction illustrated in the $\mathsf{K45_n}$ lemma, there exist $\delta_i \in R_i(\delta)$ and $\delta_{ij} \in R_j(\delta_i)$ s.t. $M^*, v \models \delta_i$ and $M^*, u \models \delta_{ij}$. By the definition of $\delta_{\tau'}$, we get that $\delta_{\tau'} = \delta_i^{\downarrow len(\tau')}$. Since $M^*, u \models \delta_{ij}$, by the definition of δ_τ and Proposition 5, we get that $\delta_\tau = \delta_{ij}^{\downarrow len(\tau)-1}$. These, together with the symmetric property (Propositions 17), imply that $\delta_{\tau'}^{\downarrow l-len(\tau')} = \delta_i^{\downarrow l} \in R_j(\delta_{ij}^{\downarrow l-2}) = R_j(\delta_\tau^{\downarrow l-len(\tau)-1})$.
3. $v \notin T$: By the definition of δ_τ, we get that $\delta_{\tau'} \in R_j(\delta_\tau)$ and hence $\delta_{\tau'}^{\downarrow l-len(\tau')} \in (R_j(\delta_\tau))^{\downarrow l-len(\tau')}$. By Proposition 7 and $len(\tau') = len(\tau) + 1$, we have $\delta_{\tau'}^{\downarrow l-len(\tau')} \in R_j(\delta_\tau^{\downarrow l-len(\tau)-1})$. □

Proposition 18. *Let (M, s) be a Kripke model where $M = \langle S, R, V\rangle$ and $s \in S$. Then, $M, s \models \nabla\Phi$ iff the following conditions hold:*

Forth *For all $t \in R_{\mathcal{A}}(s)$, there is $\phi \in \Phi$ s.t. $M, t \models \phi$;*
Back *For all $\phi \in \Phi$, there is $t \in R_{\mathcal{A}}(s)$ s.t. $M, t \models \phi$.*

Proof. The proof is similar to that of Proposition 1. □

Proposition 19. *Consider the context of a modal system L. Let $\delta \in C_k^P(\mathsf{L})$ where $k \in \mathbb{N}$ and $P \subseteq \mathcal{P}$ be finite. Let $\phi \in \mathcal{L}_{\mathbf{PC}}^{\mathbf{K}}$ s.t. $dep(\phi) \leq k$ and $\mathcal{P}(\phi) \subseteq P$. Then either $\delta \models \phi$ or $\delta \models \neg\phi$.*

Proof sketch. The proof is similar to that of Proposition 3 except we consider one more case $\phi = \mathbf{C}\phi_0$ whose proof is similar to that of the case $\phi = \mathbf{K}\phi'$. □

Proposition 20. *Consider the context of a modal system L. Let $\delta \in C_k^P(\mathsf{L})$ where $k \in \mathbb{N}$ and $P \subseteq \mathcal{P}$ is finite. Let $\phi \in \mathcal{L}_{\mathbf{PC}}^{\mathbf{K}}$ s.t. $dep'(\phi) \leq k$ and $\mathcal{P}(\phi) \subseteq P$. Then we can check if $\delta \models \phi$ recursively as follows:*

- *$\delta \models p$ iff p appears positively in $w(\delta)$;*
- *$\delta \models \neg\phi$ iff $\delta \not\models \phi$;*
- *$\delta \models \phi \wedge \psi$ iff $\delta \models \phi$ and $\delta \models \psi$;*
- *$\delta \models \phi \vee \psi$ iff $\delta \models \phi$ or $\delta \models \psi$;*
- *$\delta \models \mathbf{K}_i\phi$ iff for all $\delta' \in R_i(\delta)$, $\delta' \models \phi$.*
- *$\delta \models C\phi$ iff for all $\delta' \in R_{\mathcal{A}}(\delta)$, $\delta' \models \phi$.*

Proof. We only prove the case of common knowledge. The proofs of the other cases are similar to those for Proposition 4.

($\Leftarrow$) Suppose for all $\delta' \in R_{\mathcal{A}}(\delta)$, $\delta' \models \phi$. Let $M, s \models \delta$. We prove that $M, s \models C\phi$. Let $t \in R_{\mathcal{A}}(s)$. Then there is $\delta' \in R_{\mathcal{A}}(\delta)$ s.t. $M, t \models \delta'$. Since $\delta' \models \phi$, $M, t \models \phi$. So $M, s \models C\phi$. Thus $\delta \models C\phi$.

($\Rightarrow$) Suppose $\delta \models C\phi$. Let $\delta' \in R_{\mathcal{A}}(\delta)$. Since δ is satisfiable, there exists a model (M_0, s_0) of δ. Then there is $t \in R_{\mathcal{A}}(s_0)$ s.t. $M_0, t \models \delta'$. Since $\delta \models C\phi$, $M_0, t \models \phi$. Since δ' is a minterm of P and $\mathcal{P}(\phi) \subseteq P$, we have $\delta' \models \phi$. □

Proposition 21. *Let (M, s) be a Kripke model and $k \in \mathbb{N}$. Let $P \subseteq \mathcal{P}$ be finite. Then, there exists a unique $\delta \in C_k^P$ s.t. $M, s \models \delta$.*

Proposition 22. *Let (M, T) be a multi-pointed model and $k \in \mathbb{N}$. Let $P \subseteq \mathcal{P}$ be finite. Then, there exists a unique set $\Phi \subseteq C_k^P$ s.t. (M, T) is Φ-complete.*

Proof. Here we show Propositions 21 and 22 together.

Similarly to the proof of Propositions 5 and 14, we get that "if Proposition 21 holds for the case C_k^P, then Proposition 22 also holds for the same case".

Now, we prove that Proposition 21 holds for the base case. The proof of the induction step is similar to that of Proposition 5. By Propositions 5 and 14, there exists a unique $\delta_0 \in E_0^P$ s.t. $M, s \models \delta_0$; and there exists a unique $\Phi_0 \subseteq E_0^P$ s.t. $(M, R_{\mathcal{A}}(s))$ is Φ_0-complete. We define $\delta = \delta_0 \wedge \nabla\Phi_0$. Obviously, $M, s \models \delta$. Similarly to the proof of uniqueness of the induction step of Proposition 5, we can get that δ is the unique depth 0 pc-canonical formula of (M, s). □

Proposition 23. *Consider the context of a modal system* L. *Let* $\phi \in \mathcal{L}_{\mathbf{PC}}^{\mathbf{K}}$, $k \geq dep'(\phi)$ *and* $P = \mathcal{P}(\phi)$. *Then there exists a unique set* $\Phi \subseteq C_k^P(\mathsf{L})$ *s.t.* $\phi \equiv \bigvee \Phi$.

Proof sketch. The proof is the same as that of Proposition 6 except we use $C_k^P(\mathsf{L})$ and Propositions 3 and 5 instead of $E_k^P(\mathsf{L})$ and Propositions 19 and 21 respectively. □

Proposition 24. *Let* δ *be a pc-canonical formula and* $l < dep'(\delta)$. *Then for all* $i \in \mathcal{A}$, *we have* $R_i(\delta^{\downarrow l}) = (R_i(\delta))^{\downarrow l}$.

Proof sketch. The proof is the same as that of Proposition 7 except we use $dep'(\delta)$ instead of $dep(\delta)$. □

Proposition 25. *Let* δ *be a pc-canonical formula and* p *an atom. Then,* $\delta \models \delta^p$.

Proof sketch. The proof is similar to that of Proposition 12. □

Proposition 26. *Let* δ *be a satisfiable pc-canonical formula where* $dep'(\delta) \geq 1$. *Then*

$$R_{\mathcal{A}}(\delta) = \bigcup_{i \in \mathcal{A}} \bigcup_{\delta_i \in R_i(\delta)} [\{w(\delta_i)\} \cup R_{\mathcal{A}}(\delta_i)].$$

Proof. We use Φ to denote the right hand side of the equation. Let (M, s) be a model of δ. Any world reachable from s is either an i-child of s, or a world reachable from an i-child of s for some $i \in \mathcal{A}$, *i.e.*, $R_{\mathcal{A}}(s) = \bigcup_{i \in \mathcal{A}} \bigcup_{t \in R_i(s)} [\{t\} \cup R_{\mathcal{A}}(t)]$, which we denote by T. Obviously, $(M, R_{\mathcal{A}}(s))$ is $R_{\mathcal{A}}(\delta)$-complete, and (M, T) is Φ-complete. Since $R_{\mathcal{A}}(s) = T$, (M, T) is also $R_{\mathcal{A}}(\delta)$-complete. So both $R_{\mathcal{A}}(\delta)$ and Φ are the depth 0 pc-canonical formula set of (M, T) (Proposition 22). Hence $R_{\mathcal{A}}(\delta) = \Phi$. □

Proposition 27. *Let* δ *be a* $\mathsf{T_nPC}$ *satisfiable pc-canonical formula where* $dep'(\delta) \geq 1$. *Let* $l \in \mathbb{N}$ *s.t.* $1 \leq l \leq dep(\delta)$. *Then, for every* $i \in \mathcal{A}$, *we have* $\delta^{\downarrow l} \in R_i(\delta^{\downarrow l-1})$.

Proof sketch. The proof is the same as that of Proposition 13 except we use C_{k-1}^P and Proposition 21 instead of E_{k-1}^P and Proposition 5 respectively. □

Proposition 28. *Let* δ *be a* $\mathsf{T_nPC}$ *satisfiable pc-canonical formula. Then, we have* $w(\delta) \in R_{\mathcal{A}}(\delta)$.

Proof. The proof is similar to that of Proposition 13. □

Proposition 29. *Let* δ *be a* $\mathsf{K45_nPC}$ *satisfiable pc-canonical formula, where* $dep'(\delta) \geq 2$. *Let* $l \in \mathbb{N}$ *s.t.* $l < dep'(\delta)$. *Then, for all* $i \in \mathcal{A}$ *and* $\delta_i \in R_i(\delta)$, $(R_i(\delta))^{\downarrow l} = R_i(\delta_i^{\downarrow l-1})$.

Proof sketch. The proof is the same as that of Proposition 16 except we use Proposition 14 instead of Proposition 22. □

Proposition 30. *Let* δ *be a* $\mathsf{K45_nPC}$ *satisfiable pc-canonical formula where* $dep'(\delta) \geq 1$. *Then, for all* $i \in \mathcal{A}$ *and* $\delta_i \in R_i(\delta)$, *we have* $R_{\mathcal{A}}(\delta_i) = \bigcup_{\delta_i' \in R_i(\delta)} [\{w(\delta_i')\} \cup R_{\mathcal{A}}(\delta_i')]$.

Proof. Let (M, s) be a model of δ. There exists a world $t \in R_i(s)$ s.t. $M, t \models \delta_i$. Let $v \in R_{\mathcal{A}}(t)$. Since M is transitive and Euclidean, we have $t \in R_i(t')$ and $t' \in R_i(t)$ for all $t' \in R_i(s)$. If $v \in R_i(t)$, then v is an i-child of t. Otherwise, we get that v is a descendant of an i-child t' of s since $t' \in R_i(s)$. Thus $R_{\mathcal{A}}(t) = \bigcup_{t' \in R_i(s)} [\{t'\} \cup R_{\mathcal{A}}(t')]$.

The rest of the proof is similar to that of the transitive closure property (Proposition 26). □

Proposition 31. *Let* δ *be a* $\mathsf{K45_nPC}$ *satisfiable pc-canonical formula where* $dep'(\delta) \geq 1$. *Then, for all* $i \in \mathcal{A}$ *and* $\delta_i, \delta_i' \in R_i(\delta)$, *we have* $R_{\mathcal{A}}(\delta_i) = R_{\mathcal{A}}(\delta_i')$.

Proof. By Proposition 30, we have $R_{\mathcal{A}}(\delta_i) \subseteq R_{\mathcal{A}}(\delta_i')$ and $R_{\mathcal{A}}(\delta_i) \supseteq R_{\mathcal{A}}(\delta_i')$. Hence, $R_{\mathcal{A}}(\delta_i) = R_{\mathcal{A}}(\delta_i')$. □

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