Incorporating Action Models into the Situation Calculus

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Abstract While both situation calculus and dynamic epistemic logics (DELs) are concerned with reasoning about actions and their effects, historically, the emphasis of situation calculus was on physical actions in the single-agent case, in contrast, DELs focused on epistemic actions in the multi-agent case. In recent years, cross-fertilization between the two areas has begun to attract attention. In this paper, we incorporate the idea of action models from DELs into the situation calculus to develop a general multi-agent extension of it. We analyze properties of beliefs in this extension, and prove that action model logic can be embedded into the extended situation calculus. Examples are given to illustrate the modeling of multi-agent scenarios in the situation calculus.

Key words: The situation calculus, Dynamic epistemic logics, Reasoning about actions and change, Action models, Knowledge and belief

1 Introduction

While both situation calculus [19] and dynamic epistemic logics (DELs) [10] are concerned with reasoning about actions and their effects, historically, the emphasis of situation calculus was on physical actions in the single-agent case, in contrast, DELs focused on epistemic actions in the multi-agent case. In recent years, cross-fertilization between the two areas has begun to attract attention. In particular, van Benthem [7] proposed the idea that situation calculus and modal logic meet and

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merge. Van Ditmarsch *et al.* [11] embedded a propositional fragment of the situation calculus into a DEL. Kelly and Pearce [12] incorporated ideas from DELs to handle regression for common knowledge in the situation calculus. Baral [5] proposed to combine results from reasoning about actions and DELs.

In a multi-agent setting, the agents in the domain may have different perspectives of the actions. Baltag *et al.* [4] introduced a construct called an action model to represent these differences of perspectives. An action model consists of a set of actions, a precondition for each action, and a binary relation on the set of actions for each agent, which represents the agent's ability to distinguish between the actions. Moreover, they defined an operation by which an action model may be used to update a Kripke world to obtain a successor world modeling the effects of the action execution. They proposed a logic, called action model logic, to reason about action models and their effects on agents' epistemic states. Van Benthem *et al.* [8] generalized the concept of action model to that of update model where each action is also associated with a postcondition. So action models can model events which bring about epistemic change, but update models can model events which can not only change agents' epistemic states but also the world state.

The situation calculus was first introduced by McCarthy and Hayes [16], and historically, one of its major concerns was how to solve the frame problem, that is, how to represent the effects of a world-changing action without explicitly specifying which conditions are not affected by the action. Reiter [18] gave a solution to the frame problem under some conditions in the form of successor state axioms. This solution to the frame problem has proven useful as the foundation for the high-level robot programming language Golog [15]. Scherl and Levesque [20, 21] extended Reiter's solution to cover epistemic actions in the single-agent case. Later, Shapiro *et al.* [22] extended their work to the multi-agent case, but they only considered public actions whose occurrence is common knowledge. In the last decade, Lakemeyer and Levesque [13, 14] proposed a logic called ES, which is a fragment of the situation calculus with knowledge. Recently, Belle and Lakemeyer [6] gave a multi-agent extension of ES, but as [22], they only considered public actions. So up to now, although there have been extensions of the situation calculus into the multi-agent case, they are not able to account for arbitrary multi-agent scenarios.

In this paper, we incorporate action models into the situation calculus to develop a general multi-agent extension of it. We analyze properties of beliefs in this extension, and prove that action model logic can be embedded into the extended situation calculus. Examples are given to illustrate the modeling of multi-agent scenarios in the situation calculus.

The rest of the paper is organized as follows. In the next section, we introduce the situation calculus and action model logic. In Section 3, we present a multi-agent extension of the situation calculus by incorporating action models. Section 4 analyzes properties of beliefs in the extended situation calculus, and Section 5 shows that action model logic can be embedded into the multi-agent situation calculus. In Section 6, we present two extended examples of modeling multi-agent scenarios in the situation calculus. Finally, we conclude and describe some future work.

2 Preliminaries

In this section, we introduce the situation calculus, and action model logic.

2.1 The situation calculus and Golog

The situation calculus [19] is a many-sorted first-order language suitable for describing dynamic worlds. There are three disjoint sorts: $action$ for actions, $situation$ for situations, and $object$ for everything else. A situation calculus language $\mathscr{L}_{sc}$ has the following components: a constant S_0 denoting the initial situation; a binary function $do(a,s)$ denoting the successor situation to s resulting from performing action a; a binary predicate $s \sqsubset s'$ meaning that situation s is a proper subhistory of situation s'; a binary predicate $Poss(a,s)$ meaning that action a is possible in situation s; action functions; a finite number of relational and functional fluents, *i.e.*, predicates and functions taking a situation term as their last argument; and a finite number of situation-independent predicates and functions.

The situation calculus has been extended to accommodate sensing and knowledge. Assume that in addition to ordinary actions that change the world, there are sensing actions which do not change the world but tell the agent information about the world. A special binary function $SR(a,s)$ is used to characterize what the sensing action tells the agent about the world. Knowledge is modeled in the possible-world style by introducing a special fluent $K(s',s)$, meaning that situation s' is accessible from situation s. Note that the order of the arguments is reversed from the usual convention in modal logic. Then knowing ϕ at situation s is represented as follows:

$$\mathbf{Knows}(\phi(now),s) \stackrel{def}{=} \forall s'.K(s',s) \supset \phi(s'),$$

where *now* is used as a placeholder for a situation argument. For example,

$$\mathbf{Knows}(\exists s^*.now = do(open,s^*),s)$$

means knowing that the *open* action has just been executed. When "*now*" only appears as a situation argument to fluents, it is often omitted.

Scherl and Levesque [20] proposed the following successor state axiom for the K fluent: (Throughout this paper, free variables are assumed to be universally quantified from outside.)

$$K(s',do(a,s)) \equiv \exists s^*.K(s^*,s) \wedge s' = do(a,s^*) \wedge SR(a,s^*) = SR(a,s).$$

Intuitively, situation s' is accessible after action a is done in situation s iff it is the result of doing a in some s^* which is accessible from s and agrees with s on the sensing result.

Based on the situation calculus, a logic programming language Golog [15] has been designed for high-level robotic control. The formal semantics of Golog is specified by an abbreviation $Do(\delta,s,s')$, which intuitively means that executing δ brings us from situation s to s'. It is inductively defined on δ as follows, where we omit the definition of procedures:

1. Primitive actions:
   $Do(\alpha,s,s') \stackrel{def}{=} Poss(\alpha,s) \wedge s' = do(\alpha,s)$.
2. Test actions:
   $Do(\phi?,s,s') \stackrel{def}{=} \phi[s] \wedge s = s'$.
   Here, ϕ is a situation-suppressed formula, *i.e.*, a situation calculus formula with all situation arguments suppressed, and $\phi[s]$ denotes the formula obtained from ϕ by taking s as the situation arguments of all fluents mentioned in ϕ.
3. Sequence:
   $Do(\delta_1;\delta_2,s,s') \stackrel{def}{=} (\exists s'').Do(\delta_1,s,s'') \wedge Do(\delta_2,s'',s')$.
4. Nondeterministic choice of two actions:
   $Do(\delta_1 \mid \delta_2,s,s') \stackrel{def}{=} Do(\delta_1,s,s') \vee Do(\delta_2,s,s')$.
5. Nondeterministic choice of action arguments: Execute $\delta(x)$ with a nondeterministically chosen argument x.
   $Do((\pi x)\delta(x),s,s') \stackrel{def}{=} (\exists x)Do(\delta(x),s,s')$.
6. Nondeterministic iteration: Execute δ zero or more times.
   $Do(\delta^*,s,s') \stackrel{def}{=} (\forall P).\{(\forall s_1)P(s_1,s_1) \wedge$
   $(\forall s_1,s_2,s_3)[P(s_1,s_2) \wedge Do(\delta,s_2,s_3) \supset P(s_1,s_3)]\} \supset P(s,s')$.

Conditionals and loops are defined as abbreviations:

if ϕ **then** δ_1 **else** δ_2 **fi** $\stackrel{def}{=} [\phi?;\delta_1] \mid [\neg\phi?;\delta_2]$,

while ϕ **do** δ **od** $\stackrel{def}{=} [\phi?;\delta]^*;\neg\phi?$.

For example, the following is a Golog program which nondeterministically moves a block onto another, so long as there are at least two blocks on the table:

$$\textbf{while } (\exists x,y)[ontable(x) \wedge ontable(y) \wedge x \neq y] \textbf{ do}$$
$$(\pi u,v)move(u,v) \textbf{ od}$$

2.2 Action model logic (AML)

In a nutshell, action model logic (AML) extends epistemic logic with reasoning about epistemic actions which bring about epistemic change. We now present the syntax and semantics of action model logic. We fix a finite set of agents $\mathscr{A}$ and a countable set of propositional atoms $\mathscr{P}$. We first define Kripke models.

Definition 1. A Kripke model M is a triple (S,R,V) where

- S is a set of states;

- For each agent i, R_i is a binary relation on S;
- For each $t \in S$, $V(t)$ is a subset of the atoms.

A pointed Kripke model is a pair (M, s_0) where M is a Kripke model and s_0 is a state of M.

Intuitively, a Kripke model represents agents' uncertainty about the current world state. Here, S is the set of all possible world states; $p \in V(t)$ means that proposition p is true in state t; and tR_it' means that in state t, agent i thinks that t' might be the actual state.

An action model is a Kripke model of "actions", which represents the agents' uncertainty about the current action. The definition of action models is similar to that of Kripke models except: a truth assignment is associated to each state in a Kripke model, but a precondition is associated to each action in an action model.

Definition 2. An action model over a language $\mathscr{L}$ is a triple $(A, \rightarrow, pre)$ where

- A is a set of action points;
- For each agent i, $\rightarrow_i$ is a binary relation on A;
- For each action point e, $pre(e) \in \mathscr{L}$ is its precondition.

A pointed action model is a pair (N, e_0) where N is an action model and e_0 is an action point of N.

Given a Kripke model and an action model, by the product update operation, defined as follows, we obtain the new Kripke model resulting from executing the action model in the given Kripke model.

Definition 3. Let $M = (S, R, V)$ be a Kripke model, and $t_0 \in S$. Let $N = (A, \rightarrow, pre)$ be an action model, and $e_0 \in A$ such that $M, t_0 \models pre(e_0)$. The product of (M, t_0) and (N, e_0), denoted by $(M, t_0) \otimes (N, e_0)$, is a pointed Kripke model (M', t_0') where $M' = (S', R', V')$, and

- $S' = \{(t, e) \mid t \in S, e \in A, \text{ and } M, t \models pre(e)\}$;
- $t_0' = (t_0, e_0)$;
- $(t, e) R_i' (t', e')$ iff tR_it' and $e \rightarrow_i e'$;
- For each $(t, e) \in S'$, $V'((t, e)) = V(t)$.

Intuitively, (t, e) is the world state resulting from executing action e in state t. Note that e is an epistemic action: it does not change the world state, thus the truth assignment associated to (t, e) is the same as that associated to t. In state (t, e), agent i considers (t', e') as a possible state if she considers t' as a possible alternative of t and e' as a possible alternative of e.

The language of action model logic extends the language of epistemic logic with a construct $[N, e_0]\phi$, which intuitively means that formula ϕ holds after the execution of the pointed action model (N, e_0).

Definition 4. The language $\mathscr{L}_{am}$ of action model logic is recursively defined as follows:

1. Any propositional atom $p \in \mathscr{P}$ is an AML formula.
2. If ϕ and ψ are AML formulas, so are $\neg\phi$ and $(\phi \wedge \psi)$.
3. If ϕ is an AML formula, so are $B_i\phi$ and $C_{\mathscr{E}}\phi$, where $i \in \mathscr{A}$, $\mathscr{E} \subseteq \mathscr{A}$.
4. If ϕ is an AML formula and (N, e_0) is a pointed action model with a finite domain and such that for all action points e, $pre(e)$ is an AML formula, so is $[N, e_0]\phi$.

The following is a complexity measure of AML formulas as presented in [10]:

Definition 5. The complexity measure $c : \mathscr{L}_{am} \to \mathbb{N}$ is inductively defined as follows:

1. $c(p) = 1$;
2. $c(\neg\phi) = 1 + c(\phi)$;
3. $c(\phi \wedge \psi) = 1 + \max\{c(\phi), c(\psi)\}$;
4. $c(B_i\phi) = 1 + c(\phi)$;
5. $c(C_{\mathscr{E}}\phi) = 1 + c(\phi)$;
6. $c([N, e_0]\phi) = (4 + \max\{c(pre(e)) \mid e \text{ is an action point of } N\}) \cdot c(\phi)$.

We now present the semantics of action model logic:

Definition 6. Let $M = (S, R, V)$ be a Kripke model and t_0 a state of M. We interpret the formulas by induction on their complexity as follows:

1. $M, t_0 \models p$ iff $p \in V(t_0)$;
2. $M, t_0 \models \neg\phi$ iff $M, t_0 \not\models \phi$;
3. $M, t_0 \models \phi \wedge \psi$ iff $M, t_0 \models \phi$ and $M, t_0 \models \psi$;
4. $M, t_0 \models B_i\phi$ iff for all t such that $t_0 R_i t$, $M, t \models \phi$;
5. $M, t_0 \models C_{\mathscr{E}}\phi$ iff for all t such that $t_0 R_{\mathscr{E}} t$, $M, t \models \phi$, where $R_{\mathscr{E}}$ is the reflexive transitive closure of the union of R_i for $i \in \mathscr{E}$;
6. $M, t_0 \models [N, e_0]\phi$ iff if $M, t_0 \models pre(e_0)$, then $(M, t_0) \otimes (N, e_0) \models \phi$.

A formula ϕ is valid if it is true in any pointed Kripke model.

We end this section with an example:

Example 1. [10] Two stockbrokers Ann and Bob are having a little break in a Wall Street bar, sitting at a table. A messenger comes in and delivers a letter to Ann. On the envelope is written "urgently requested data on United Agents". Let atom p mean that "United Agents is doing well". Consider the following scenarios:

1. Bob sees that Ann reads the letter. From Bob's point of view, Ann could learn p or she could learn $\neg p$, and he cannot distinguish between these two actions. But Ann can certainly distinguish between them. Thus we get the following action model: $read = (A, \to, pre)$, where $A = \{0, 1\}$, $pre(0) = \neg p$, $pre(1) = p$, $\to_a$ is the identity relation, and $\to_b$ is the total relation.
2. Bob leaves the table; Ann may have read the letter while Bob is away. From Bob's point of view, there are 3 possibilities: Ann learns p, Ann learns $\neg p$, and Ann learns nothing, and he cannot distinguish between these actions. Thus the action model is: $mayread = (A, \to, pre)$, where $A = \{0, 1, t\}$, $pre(0) = \neg p$, $pre(1) = p$, $pre(t) = true$, $\to_a$ is the identity relation, and $\to_b$ is the total relation.

3 A multi-agent extension of the situation calculus

In this section, we present a multi-agent extension of the situation calculus by incorporating action models. Instead of Scherl and Levesque's K fluent, we now use a fluent $B(i,s',s)$, which means that agent i considers situation s' accessible from situation s. We introduce a special predicate $A(i,a',a,s)$, meaning that in situation s, agent i considers action a' as a possible alternative of action a.

We assume that there are two types of primitive actions: ordinary actions which change the world, and epistemic actions which do not change the world but informs the agent. We use the action precondition axiom to specify what the epistemic action tells the agent about the current situation. For example, we may have an epistemic action $ison(i,x)$ which tells agent i that switch x is on. This is axiomatized as:

$$Poss(ison(i,x),s) \equiv on(x,s).$$

In particular, there is a special epistemic action nil, meaning that nothing happens, with the axiom $Poss(nil,s) \equiv true$. Note that a sensing action which tells the agent whether ϕ holds can be treated as the nondeterministic choice of two epistemic actions: one is possible iff ϕ holds, and the other is possible iff $\neg\phi$ holds.

We propose the following successor state axiom for the B fluent:

$$B(i,s',do(a,s)) \equiv \exists s^* \exists a^*.B(i,s^*,s) \wedge A(i,a^*,a,s) \wedge \\ (Poss(a,s) \supset Poss(a^*,s^*)) \wedge s' = do(a^*,s^*).$$

Intuitively, for agent i, situation s' is accessible after action a is performed in situation s iff it is the result of doing some alternative a^* of a in some s^* accessible from s, and executability of a in s implies that of a^* in s^*. Note that when a is not possible in s, we do not care whether a^* is possible in s^*.

In the multi-agent case, a domain of application is specified by a basic action theory of the form:

$$\mathscr{D} = \Sigma \cup \mathscr{D}_{ap} \cup \mathscr{D}_{ss} \cup \mathscr{D}_{aa} \cup \mathscr{D}_{una} \cup \mathscr{D}_{S_0}, \text{ where}$$

1. Σ are the foundational axioms:

(F1) $do(a_1,s_1) = do(a_2,s_2) \supset a_1 = a_2 \wedge s_1 = s_2$
(F2) $(\neg s \sqsubset S_0) \wedge (s \sqsubset do(a,s') \equiv s \sqsubseteq s')$
(F3) $\forall P.\forall s[Init(s) \supset P(s)] \wedge \forall a,s[P(s) \supset P(do(a,s))] \supset (\forall s)P(s)$, where
$Init(s) \stackrel{def}{=} \neg(\exists a,s')s = do(a,s')$.
(F4) $B(i,s,s') \supset [Init(s) \equiv Init(s')]$.

Intuitively, $Init(s)$ means s is an initial situation. A model of $\{F1,F2,F3\}$ consists of a forest of isomorphic trees rooted at the initial situations. F4 specifies that initial situations can be B-related to only initial situations.

2. $\mathcal{D}_{ap}$ is a set of action precondition axioms, one for each action function C, of the form $Poss(C(\mathbf{x}), s) \equiv \Pi_C(\mathbf{x}, s)$. This includes the precondition axioms for epistemic actions.
3. $\mathcal{D}_{ss}$ is a set of successor state axioms (SSAs) for fluents, one for each fluent F, of the form $F(\mathbf{x}, do(a, s)) \equiv \Phi_F(\mathbf{x}, a, s)$. This includes the SSA for the B fluent. The SSAs for ordinary fluents must satisfy the no-side-effect conditions, *i.e.*, they are not affected by epistemic actions.
4. $\mathcal{D}_{aa}$ is a set of action alternative axioms, one for each action function C, of the form $A(i, a, C(\mathbf{x}), s) \equiv \Psi_C(i, a, \mathbf{x}, s)$.
5. $\mathcal{D}_{una}$ is the set of unique names axioms for actions:

$$C(\mathbf{x}) \neq C'(\mathbf{y}), \text{ and } C(\mathbf{x}) = C(\mathbf{y}) \supset \mathbf{x} = \mathbf{y},$$

   where C and C' are distinct action functions.
6. $\mathcal{D}_{S_0}$ is a set of sentences about S_0.

In the rest of the paper, when we present a basic action theory, we will only present relevant axioms from $\mathcal{D}_{ap} \cup \mathcal{D}_{ss} \cup \mathcal{D}_{aa} \cup \mathcal{D}_{S_0}$.

Example 2. Consider a simple blocks world. There is a single physical action: $move(x, y)$, moving block x onto block y. There are two fluents: $clear(x, s)$, block x has no blocks on top of it; $on(x, y, s)$, block x is on block y. The following are the action precondition and successor state axioms:

$$
\begin{aligned}
&Poss(move(x, y), s) \equiv clear(x, s) \wedge clear(y, s) \wedge x \neq y \\
&on(x, y, do(a, s)) \equiv a = move(x, y) \vee \\
&\qquad on(x, y, s) \wedge \neg(\exists z)a = move(x, z), \\
&clear(x, do(a, s)) \equiv (\exists y)(\exists z)a = move(y, z) \wedge on(y, x, s) \vee \\
&\qquad clear(x, s) \wedge \neg(\exists y)a = move(y, x).
\end{aligned}
$$

We now axiomatize in the situation calculus the letter example of Section 2.2.

Example 3.

1. Bob sees that Ann reads the letter. We introduce an epistemic action $read(e)$, which means that Ann senses the truth value of p with result e while Bob is observing her. The axioms are:

$$
\begin{aligned}
&Poss(read(e), s) \equiv (e = 1 \equiv p(s)), \\
&A(i, a, read(e), s) \equiv (i = ann \supset a = read(e)) \wedge (i = bob \supset \exists e'.a = read(e')).
\end{aligned}
$$

   So Ann can distinguish between $read(1)$ and $read(0)$, but Bob can't.
2. Bob thinks Ann may have read the letter. We introduce an epistemic action $mread(e)$, which means that Ann senses the truth value of p with result e while Bob is not sure about whether this happens. The axioms are:

$$\begin{aligned}
&Poss(mread(e),s) \equiv (e=1 \equiv p(s)),\\
&A(i,a,mread(e),s) \equiv (i=ann \supset a=mread(e)) \wedge\\
&\qquad\qquad (i=bob \supset a=nil \vee \exists e'.a=mread(e')),\\
&A(i,a,nil,s) \equiv (i=ann \supset a=nil) \wedge\\
&\qquad\qquad (i=bob \supset a=nil \vee \exists e'.a=mread(e'))
\end{aligned}$$

So Bob can't distinguish between the three actions $mread(1)$, $mread(0)$, and nil.

Finally, we introduce some notation which will be used in the rest of the paper. Let $\phi(s)$ be a formula with a single situation variable s.

1. Agent i believes ϕ:

$$\mathbf{Bel}(i,\phi(now),s) \stackrel{def}{=} \forall s'.B(i,s',s) \supset \phi(s').$$

2. Agent i truly believes ϕ:

$$\mathbf{TBel}(i,\phi(now),s) \stackrel{def}{=} \phi(s) \wedge \mathbf{Bel}(i,\phi(now),s).$$

3. Agent i believes whether ϕ holds:

$$\mathbf{BW}(i,\phi(now),s) \stackrel{def}{=} \mathbf{Bel}(i,\phi(now),s) \vee \mathbf{Bel}(i,\neg\phi(now),s).$$

4. Let $\mathscr{E}$ be a subset of the agents. We let $C(\mathscr{E},s',s)$ denote the reflexive transitive closure of $\exists i \in \mathscr{E}.B(i,s',s)$, which can be defined with a second-order formula:

$$C(\mathscr{E},s',s) \stackrel{def}{=}$$
$$\forall P.\forall u P(u,u) \wedge \forall i \in \mathscr{E}, u,v,w[P(u,v) \wedge B(i,v,w) \supset P(u,w)] \supset P(s',s).$$

5. The agents commonly know ϕ:

$$\mathbf{CKnows}(\phi(now),s) \stackrel{def}{=} \forall s'.C(\mathscr{A},s',s) \supset \phi(s'),$$

where $\mathscr{A}$ is the set of all agents.

4 Properties of beliefs

In this section, we analyze properties of beliefs in our formalism. We begin with the main property of beliefs. We use $\Psi_0(a,s)$ to denote the following formula:

$$\forall i.\mathbf{Bel}(i,\exists s^*\exists a^*.A(i,a^*,a,s) \wedge now = do(a^*,s^*) \wedge Poss(a^*,s^*) \wedge \mathscr{D}_{ss}[a^*,s^*], do(a,s)),$$

where $\mathscr{D}_{ss}[a^*,s^*]$ denotes the instantiation of the SSAs for ordinary fluents wrt a^* and s^*. This says that in the situation resulting from doing action a, each agent i

believes that some alternative of a was possible and has happened. We use $\Psi_{n+1}(a,s)$ to denote the following formula:

$$\forall i.\mathbf{Bel}(i, \exists s^*\exists a^*.A(i,a^*,a,s) \wedge now = do(a^*,s^*) \wedge Poss(a^*,s^*) \wedge \mathscr{D}_{ss}[a^*,s^*] \wedge \Psi_n(a^*,s^*), do(a,s)).$$

Thus $\Psi_1(a,s)$ says that in the situation resulting from doing action a, each agent i believes that some alternative a^* of a was possible, has happened, and in the resulting situation, each agent believes that some alternative of a^* was possible and has happened. By the SSA for the B fluent, it is straightforward to prove:

Theorem 1. *For all n,* $\mathscr{D} \models \forall a \forall s.Poss(a,s) \supset \Psi_n(a,s)$.

Proof. We prove by induction on n.

Basis: $n = 0$. This directly follows from the SSA for the B fluent.

Induction step: Assume that $\mathscr{D} \models \forall a \forall s.Poss(a,s) \supset \Psi_n(a,s)$. By the SSA for the B fluent, we have

$$\forall a \forall s.Poss(a,s) \supset \forall i.\mathbf{Bel}(i, \exists s^*\exists a^*.A(i,a^*,a,s) \wedge now = do(a^*,s^*) \wedge Poss(a^*,s^*) \wedge \mathscr{D}_{ss}[a^*,s^*], do(a,s)).$$

By the induction hypothesis, we have

$$\forall a \forall s.Poss(a,s) \supset \forall i.\mathbf{Bel}(i, \exists s^*\exists a^*.A(i,a^*,a,s) \wedge now = do(a^*,s^*) \wedge Poss(a^*,s^*) \wedge \mathscr{D}_{ss}[a^*,s^*] \wedge \Psi_n(a^*,s^*), do(a,s)),$$

which is $\forall a \forall s.Poss(a,s) \supset \Psi_{n+1}(a,s)$. ■

Let $\phi(s)$ be a formula with a single situation variable s. We introduce an epistemic action $observe_\phi$, which tells the agent that ϕ holds in the current situation. The axiom is: $Poss(observe_\phi, s) \equiv \phi(s)$. It is easy to prove the following propositions. By an objective formula, we mean one which does not use the B fluent or the A predicate.

Proposition 1. *Let ϕ be an objective formula. Suppose that agent i is an* observer *of action $observe_\phi$ in situation σ, i.e.,* $\mathscr{D} \models \forall a.A(i,a,observe_\phi,\sigma) \equiv a = observe_\phi$. *Then* $\mathscr{D} \models \phi(\sigma) \supset \mathbf{Bel}(i,\phi,do(observe_\phi,\sigma))$.

Proposition 2. *Let ϕ be an objective formula. Suppose that agent i is a* partial observer *of action $observe_\phi$ in situation σ, i.e.,*
$\mathscr{D} \models \forall a.A(i,a,observe_\phi,\sigma) \equiv a = observe_\phi \vee a = observe_{\neg\phi}$.
Then $\mathscr{D} \models \phi(\sigma) \wedge \neg\mathbf{BW}(i,\phi,\sigma) \supset \neg\mathbf{BW}(i,\phi,do(observe_\phi,\sigma))$.

Proposition 3. *Let ϕ be an objective formula. Suppose that agent i is* oblivious *of action α in situation σ, i.e.,* $\mathscr{D} \models \forall a.A(i,a,\alpha,\sigma) \equiv a = nil$.
Then $\mathscr{D} \models Poss(\alpha,\sigma) \supset [\mathbf{Bel}(i,\phi,\sigma) \equiv \mathbf{Bel}(i,\phi,do(\alpha,\sigma))]$.

In the following, we show how we model some special types of actions and prove the desired properties. We first consider public sensing and reading actions: we say a sensing or reading action is public if its occurrence is common knowledge but only the performer of the action gets to know the result. The axioms are as follows:

- $sense_{\phi}(i,\mathbf{x},e)$ means agent i senses the truth value of $\phi(\mathbf{x})$ and gets result e;
- $read_f(i,\mathbf{x},y)$ means agent i reads the value of $f(\mathbf{x})$ and gets result y.

1. $Poss(read_f(i,\mathbf{x},e),s) \equiv (e = 1 \equiv \phi(\mathbf{x},s))$
2. $A(j,a,sense_{\phi}(i,\mathbf{x},e),s) \equiv \exists e'(a = sense_{\phi}(i,\mathbf{x},e')) \wedge (j = i \supset a = sense_{\phi}(i,\mathbf{x},e))$
3. $Poss(read_f(i,\mathbf{x},y),s) \equiv f(\mathbf{x},s) = y$
4. $A(j,a,read_f(i,\mathbf{x},y),s) \equiv \exists y'(a = read_f(i,\mathbf{x},y')) \wedge (j = i \supset a = read_f(i,\mathbf{x},y))$

We let $sense_{\phi}(i,\mathbf{x})$ denote $sense_{\phi}(i,\mathbf{x},1) \mid sense_{\phi}(i,\mathbf{x},0)$, and $read_f(i,\mathbf{x})$ denote $(\pi y) read_f(i,\mathbf{x},y)$. It is easy to prove:

Proposition 4. *$\mathscr{D}$ entails the following:*

1. $Do(sense_{\phi}(i,\mathbf{x}),s,s_1) \supset [B(j,s',s_1) \equiv$
   $\exists s^*.B(j,s^*,s) \wedge Do(sense_{\phi}(i,\mathbf{x}),s^*,s') \wedge (j = i \supset \phi(\mathbf{x},s) \equiv \phi(\mathbf{x},s^*))]$
2. $Do(read_f(i,\mathbf{x}),s,s_1) \supset [B(j,s',s_1) \equiv$
   $\exists s^*.B(j,s^*,s) \wedge Do(read_f(i,\mathbf{x}),s^*,s') \wedge (j = i \supset f(\mathbf{x},s) = f(\mathbf{x},s^*))]$

This is the same as Shapiro *et al.* 's extension of Scherl and Levesque's SSA for the K fluent to public sensing and reading actions in the multi-agent case [22]. So our account of beliefs and actions subsumes theirs. As an easy corollary, we get

Proposition 5. *Let ϕ be an objective formula. Then $\mathscr{D}$ entails the following:*

1. $Do(sense_{\phi}(i,\mathbf{x}),s,s_1) \supset \mathbf{BW}(i,\phi(\mathbf{x}),s_1) \wedge$
   $(j \neq i \supset \mathbf{Bel}(j,\mathbf{BW}(i,\phi(\mathbf{x})),s_1))$
2. $Do(read_f(i,\mathbf{x}),s,s_1) \supset \exists y \mathbf{Bel}(i,f(\mathbf{x}) = y,s_1) \wedge$
   $(j \neq i \supset \mathbf{Bel}(j,\exists y \mathbf{Bel}(i,f(\mathbf{x}) = y),s_1))$

Bacchus *et al.* [1] considered noisy sensors: when an agent reads the value of $f(\mathbf{x})$, she may get a value y such that $|f(\mathbf{x},s) - y| \leq b$ for some bound b. We introduce an epistemic action $nread_f(i,\mathbf{x},y)$ for this purpose, and let $nread_f(i,\mathbf{x})$ denote $(\pi y) nread_f(i,\mathbf{x},y)$. The axioms are:

1. $Poss(nread_f(i,\mathbf{x},y),a) \equiv |f(\mathbf{x},s) - y| \leq b$
2. $A(j,a,nread_f(i,\mathbf{x},y),s) \equiv (\exists y')a = nread_f(i,\mathbf{x},y') \wedge$
   $\{j = i \supset (\exists y').a = nread_f(i,\mathbf{x},y') \wedge |y - y'| \leq b\}$

As desired, we have

Proposition 6. $\mathscr{D} \models Do(nread_f(i,\mathbf{x}),s,s') \supset \exists y.\mathbf{Bel}(i,|f(\mathbf{x}) - y| \leq b,s')$.

Delgrande and Levesque [9] considered unintended actions: an agent wants to push button m, but she may push button n such that $|m - n| \leq b$. We introduce a physical action $npush(i,m,n)$, meaning that agent i wants to push button m but ends up pushing button n. We let $npush(i,m)$ denote $(\pi n) npush(i,m,n)$. The axioms are:

1. $Poss(npush(i,m,n),a) \equiv |m-n| \leq b$
2. $on(n,do(a,s)) \equiv \exists i,m.a = npush(i,m,n)$
3. $A(j,a,npush(i,m,n),s) \equiv (\exists m',n')a = npush(i,m',n') \wedge$
   $\{j = i \supset (\exists n').a = push(i,m,n')\}$

Proposition 7. $\mathscr{D} \models Do(npush(i,m),s,s') \supset \mathbf{Bel}(i,\exists n.|n-m| \leq b \wedge on(n),s')$.

Dynamic epistemic logics originated with public announcement logic, which reasons about the epistemic change brought about by public communications [17]. We have the following description for the action of publicly truthfully announcing ϕ:

1. $Poss(pub_\phi,s) \equiv \phi(s)$
2. $A(i,a,pub_\phi,s) \equiv a = pub_\phi$

Proposition 8. *Let ϕ be an objective formula. Then*
$\mathscr{D} \models \phi(s) \supset \mathbf{CKnows}(\phi,do(pub_\phi,s))$.

5 The embedding theorem

In this section, we prove that action model logic can be embedded into the extended situation calculus. We first define two functions: $\mathscr{B}$, which maps a formula ϕ in AML into a basic action theory encoding the action models involved in ϕ, and $\mathscr{F}$, which maps a formula in AML and a situation term into a formula in the situation calculus. We then prove that for any formula ϕ in AML, ϕ is valid in AML iff $\mathscr{B}(\phi) \models \mathscr{F}(\phi,S_0)$.

Definition 7. Let ϕ be a formula in AML. We define two sets $AM(\phi)$ and $Prop(\phi)$ recursively as follows:

1. $AM(\phi)$ is the set of action models N such that N appears in ϕ or there exist an action model N' which occurs in ϕ and an action point e of N' such that $N \in AM(pre(e))$;
2. $Prop(\phi)$ is the set of propositions p such that p appears in ϕ or there exist an action model N' which occurs in ϕ and an action point e of N' such that $p \in Prop(pre(e))$.

We define the vocabulary of our situation calculus language associated to ϕ as follows. For each $N \in AM(\phi)$, we introduce an action $c_N(x)$, where x ranges over the action points of N. For each $p \in Prop(\phi)$, we introduce a unary fluent $p(s)$.

Definition 8. Let ϕ be a formula in AML, and s a situation term. We define a situation calculus formula $\mathscr{F}(\phi,s)$ by induction on the complexity of ϕ as follows:

1. $\mathscr{F}(p,s) = p(s)$;
2. $\mathscr{F}(\neg\psi,s) = \neg\mathscr{F}(\psi,s)$;
3. $\mathscr{F}(\psi \wedge \eta,s) = \mathscr{F}(\psi,s) \wedge \mathscr{F}(\eta,s)$;
4. $\mathscr{F}(B_i\psi,s) = \forall s'.B(i,s',s) \supset \mathscr{F}(\psi,s')$;

5. $\mathscr{F}(C_{\mathscr{E}}\psi, s) = \forall s'.C(\mathscr{E}, s', s) \supset \mathscr{F}(\psi, s')$;
6. $\mathscr{F}([N, e_0]\psi, s) = \mathscr{F}(pre(e_0), s) \supset \mathscr{F}(\psi, do(c_N(e_0), s))$.

Note that we model the execution of a pointed action model (N, e_0) with the execution of the action $c_N(e_0)$ in the situation calculus.

Definition 9. Let ϕ be a formula in AML. Let $AM(\phi) = \{N_1, \ldots, N_m\}$, where $N_k = (A_k, \rightarrow, pre)$, $k = 1, \ldots, m$. Without loss of generality, we assume that the A_k's are pairwise disjoint. We construct a basic action theory $\mathscr{B}(\phi)$ as follows.

(A0) $e \neq e'$, where e and e' are distinct action points;
(A1) $Poss(c_{N_k}(x), s) \equiv \bigvee_{e \in A_k}[x = e \wedge \mathscr{F}(pre(e), s)]$;
(A2) $p(do(a, s)) \equiv p(s)$;
(A3) $A(j, a, c_{N_k}(x), s) \equiv \exists y.a = c_{N_k}(y) \wedge \bigvee_{(e,e') \in \rightarrow_j} x = e \wedge y = e'$;

Note that the reason we have A2 is that in action models, actions do not change the world. A3 specifies that the value of the A predicate is set according to the accessibility relations of the action models.

The following is the embedding theorem:

Theorem 2. *For any formula ϕ in AML, ϕ is valid in AML iff $\mathscr{B}(\phi) \models \mathscr{F}(\phi, S_0)$.*

To prove the embedding theorem, we first introduce a method to induce a Kripke model from a structure of the situation calculus. For a structure L and a syntactic object o, we let o^L stand for the denotation of o in L. We say that a situation is at level k if it results from performing a sequence of k actions in an initial situation. Let L be a structure of the situation calculus, τ a situation of L, and $\phi(s)$ a situation calculus formula with a free situation variable s. We use $L, \tau \models \phi(s)$ to denote that when s is interpreted as τ, L satisfies $\phi(s)$.

Definition 10. Let L be a structure of the situation calculus. Let τ be a level k situation of L. We define a Kripke model $M_L^\tau = (S, R, V)$ as follows:

- S consists of all level k situations of L;
- For any $t_1, t_2 \in S$, $t_1 R_i t_2$ iff $(i, t_2, t_1) \in B^L$;
- For each $t \in S$, $V(t)$ is the set of fluents p which are true at t in L.

We call M_L^τ the Kripke projection of L onto τ. Note that if τ_1 and τ_2 are at the same level, then $M_L^{\tau_1} = M_L^{\tau_2}$.

The proof of the embedding theorem is involved. We now explain the general idea of the proof. First, suppose $\mathscr{B}(\phi) \not\models \mathscr{F}(\phi, S_0)$. Let L be a model of $\mathscr{B}(\phi) \cup \{\neg\mathscr{F}(\phi, S_0)\}$. We show that the pointed Kripke model $(M_L^{\tau_0}, \tau_0)$ satisfies $\neg\phi$, where $\tau_0 = S_0^L$, and $M_L^{\tau_0}$ is the Kripke projection of L onto τ_0. The other direction is more complicated, and the reason is that for an action point e of an action model N, $pre(e)$ may involve action models. The precondition axiom for action C_N is defined with $\mathscr{F}(pre(e), s)$. When $pre(e)$ involves action models, $\mathscr{F}(pre(e), s)$ refers to future situations of s.

Now suppose ϕ is not valid. Let (M,t_0) be a model of $\neg\phi$. We construct a model L of $\mathscr{D} = \mathscr{B}(\phi)$ as follows. First, let $L \models \mathscr{D}_{una} \cup \{A0\}$. The initial situations of L are the states of M, and L interprets S_0 as t_0. The situations of L form a forest of isomorphic trees rooted at the initial situations, where the children of each situation one-to-one correspond to the actions. We interpret the A predicate according to A3. We interpret *Poss* and the fluents by induction on the level of situations. For initial situations, we interpret the B and the p fluents according to M. Let τ be an initial situation. For each action model N and action point e of it, we let $L,\tau \models Poss(c_N(e),s)$ iff $M,\tau \models pre(e)$. Assume we have interpreted *Poss* and all the fluents at level k situations. We interpret the fluents at level $k+1$ situations according to $\mathscr{D}_{ss}$. Let τ be a level $k+1$ situation. For each action model N and action point e of it, we let $L,\tau \models Poss(c_N(e),s)$ iff $M_L^\tau,\tau \models pre(e)$. We show that L is a model of $\mathscr{B}(\phi) \cup \{\neg\mathscr{F}(\phi,S_0)\}$.

In the above outline of proof, for the $\mathscr{L}_{sc}$-structure L we construct, we have that the Kripke projection of L onto the initial situations – $M_L^{t_0}$, is isomorphic to M. The isomorphism of Kripke models will play an important role in our proof. However, instead of requiring that two pointed Kripke models (M_1,t_1) and (M_2,t_2) be isomorphic, we only require that their reductions be isomorphic, *i.e.*, the resulting pointed Kripke models are isomorphic after we remove the states of M_i not reachable from t_i for $i = 1,2$. In the following, we define the concepts of isomorphism and reduction of Kripke models and study their basic properties.

Let (M_1,t_1) and (M_2,t_2) be two pointed Kripke models. We let $h : (M_1,t_1) \cong (M_2,t_2)$ denote that h is an isomorphism from (M_1,t_1) to (M_2,t_2), *i.e.*,

- h is a bijection from the states of M_1 to those of M_2, and $h(t_1) = t_2$;
- h preserves the accessibility relation, *i.e.*, for any agent i and any two states t and t' of M_1, tR_it' iff $h(t)R_ih(t')$;
- h preserves the atoms, that is, for any atom p and state t of M_1, p holds at t iff p holds at $h(t)$.

Let (M,t_0) be a pointed Kripke model, we use $\mathscr{R}(M,t_0)$ to denote the pointed Kripke model (M',t_0), where M' is obtained from M by removing those states not reachable from t_0. It is easy to prove the following properties:

Proposition 9. *Let $h : \mathscr{R}(M_1,t_1) \cong \mathscr{R}(M_2,t_2)$, and t a state of M_1 reachable from t_1. Then $\mathscr{R}(M_1,t) \cong \mathscr{R}(M_2,h(t))$.*

Proof. Since t is reachable from t_1, the states of $\mathscr{R}(M_1,t)$ are contained in those of $\mathscr{R}(M_1,t_1)$. It suffices to prove that for any state t' of M_1 reachable from t, $h(t')$ is reachable from $h(t)$, which holds because h preserves the accessibility relation. ■

Proposition 10. *Let $h : \mathscr{R}(M_1,t_1) \cong \mathscr{R}(M_2,t_2)$. Then for any AML formula ϕ, $M_1,t_1 \models \phi$ iff $M_2,t_2 \models \phi$.*

Proof. See Appendix.

In the above outline of proof, the $\mathscr{L}_{sc}$-structure L we construct is a model of $\mathscr{D} - \mathscr{D}_{ap}$ and it has a property defined as follows:

Definition 11. We say that an $\mathscr{L}_{sc}$-structure L has property C1 if for any situation τ, action model N and action point e of N, $L,\tau \models Poss(c_N(e),s)$ iff $M_L^{\tau},\tau \models pre(e)$.

In the following, we study properties of models of $\mathscr{D}-\mathscr{D}_{ap}$ with C1. We first prove a proposition which shows that the execution of a pointed action model (N,e_0) can be modeled in the situation calculus with the execution of the action $c_N(e_0)$.

Proposition 11. *Let L be a model of $\mathscr{D}-\mathscr{D}_{ap}$ with C1, and τ_0 a situation of L. Let (M,t_0) be a pointed Kripke model, and (N,e_0) a pointed action model. Suppose that $h:\mathscr{R}(M,t_0)\cong\mathscr{R}(M_L^{\tau_0},\tau_0)$ and $M,t_0\models pre(e_0)$. Then $\mathscr{R}((M,t_0)\otimes(N,e_0))\cong\mathscr{R}(M_L^{\tau_1},\tau_1)$ where τ_1 is $do^L(c_N(e_0)^L,\tau_0)$.*

Proof. See Appendix.

Next, we prove a lemma which shows that an AML formula ϕ can be modeled by the situation calculus formula $\mathscr{F}(\phi,s)$:

Lemma 1. *Let L be a model of $\mathscr{D}-\mathscr{D}_{ap}$ with C1, and τ_0 a situation of L. Suppose $h:\mathscr{R}(M,t_0)\cong\mathscr{R}(M_L^{\tau_0},\tau_0)$. Then $M,t_0\models\phi$ iff $L,\tau_0\models\mathscr{F}(\phi,s)$.*

Proof. We prove by induction on the complexity of ϕ.

1. ϕ is p. Then $M,t_0\models p$ iff p is true at t_0 in M iff p is true at τ_0 in L (since $\mathscr{R}(M,t_0)\cong\mathscr{R}(M_L^{\tau_0},\tau_0)$) iff $L,\tau_0\models p(s)$.
2. ϕ is $\neg\psi$. Then $M,t_0\models\neg\psi$ iff $M,t_0\not\models\psi$ iff $L,\tau_0\not\models\mathscr{F}(\psi,s)$ (by induction hypothesis) iff $L,\tau_0\models\neg\mathscr{F}(\psi,s)$, *i.e.*, $L,\tau_0\models\mathscr{F}(\neg\psi,s)$.
3. ϕ is $\psi\wedge\psi'$. Similar to the above case.
4. ϕ is $B_i\psi$. Let t be a state of M such that t_0R_it. By Proposition 9, $\mathscr{R}(M,t)\cong\mathscr{R}(M_L^{\tau_0},h(t))$. By induction hypothesis, $M,t\models\psi$ iff $L,h(t)\models\mathscr{F}(\psi,s')$. So $M,t_0\models B_i\psi$ iff for every t such that t_0R_it, $M,t\models\psi$ iff for every τ such that $(i,\tau,\tau_0)\in B^L$, $L,\tau\models\mathscr{F}(\psi,s')$ iff $L,\tau_0\models\forall s'.B(i,s',s)\supset\mathscr{F}(\psi,s')$, *i.e.*, $L,\tau_0\models\mathscr{F}(B_i\psi,s)$.
5. ϕ is $C_{\mathscr{E}}\psi$. Similar to the above case.
6. ϕ is $[N,e_0]\psi$. By induction hypothesis, $M,t_0\models pre(e_0)$ iff $L,\tau_0\models\mathscr{F}(pre(e_0),s)$. By Proposition 11, if $M,t_0\models pre(e_0)$, then $(M,t_0)\otimes(N,e_0)\cong M_L^{\tau_1},\tau_1$, where $\tau_1=do(c_N(e_0)^L,\tau_0)$. By induction hypothesis, $(M,t_0)\otimes(N,e_0)\models\psi$ iff $L,\tau_1\models\mathscr{F}(\psi,s)$. Thus $M,t_0\models[N,e_0]\psi$ iff if $M,t_0\models pre(e_0)$ then $(M,t_0)\otimes(N,e_0)\models\psi$ iff if $L,\tau_0\models\mathscr{F}(pre(e_0),s)$ then $L,\tau_1\models\mathscr{F}(\psi,s)$ iff $L,\tau_0\models\mathscr{F}(pre(e_0),s)\supset\mathscr{F}(\psi,do(c_N(e_0),s))$, which is $\mathscr{F}([N,e_0]\psi,s)$. ■

The above lemma requires that L be a model of $\mathscr{D}-\mathscr{D}_{ap}$ with C1. The lemma below shows that we can replace this requirement with the one that L is a model of $\mathscr{D}$.

Lemma 2. *Let L be a model of $\mathscr{D}$, and τ_0 a situation of L. Suppose $h:\mathscr{R}(M,t_0)\cong\mathscr{R}(M_L^{\tau_0},\tau_0)$. Then $M,t_0\models\phi$ iff $L,\tau_0\models\mathscr{F}(\phi,s)$.*

Proof. We prove by induction on the complexity of ϕ. Assume that the statement holds for all formulas less complex than ϕ. Then for any situation τ and action point e of action model N, since $pre(e)$ is less complex than ϕ and $\mathscr{R}(M_L^\tau, \tau) \cong \mathscr{R}(M_L^\tau, \tau)$, we have that $M_L^\tau, \tau \models pre(e)$ iff $L, \tau \models \mathscr{F}(pre(e), s)$. By $\mathscr{D}_{ap}$, $L, \tau \models Poss(c_N(e), s) \equiv \mathscr{F}(pre(e), s)$. Thus we have $L, \tau \models Poss(c_N(e), s)$ iff $M_L^\tau, \tau \models pre(e)$. So L satisfies C1. By applying Lemma 1, we have $M, t_0 \models \phi$ iff $L, \tau_0 \models \mathscr{F}(\phi, s)$. ■

Finally, we are ready to prove the embedding theorem:

Proof. First, suppose $\mathscr{B}(\phi) \not\models \mathscr{F}(\phi, S_0)$. Let L be a model of $\mathscr{B}(\phi) \cup \{\neg\mathscr{F}(\phi, S_0)\}$. Let $\tau_0 = S_0^L$. Since $\mathscr{R}(M_L^{\tau_0}, \tau_0) \cong \mathscr{R}(M_L^{\tau_0}, \tau_0)$, by Lemma 2, $M_L^{\tau_0}, \tau_0 \models \phi$ iff $L \models \mathscr{F}(\phi, S_0)$. Thus $M_L^{\tau_0}, \tau_0 \models \neg\phi$. So ϕ is not valid.

Now suppose ϕ is not valid. Let (M, t_0) be a model of $\neg\phi$. We construct a model L of $\mathscr{D} = \mathscr{B}(\phi)$ as follows. First, let $L \models \mathscr{D}_{una} \cup \{A0\}$. The initial situations of L are the states of M, and L interprets S_0 as t_0. The situations of L form a forest of isomorphic trees rooted at the initial situations, where the children of each situation one-to-one correspond to the actions. Thus L satisfies the foundational axioms F1, F2, and F3. We interpret the A predicate according to A3. We now interpret *Poss* and the fluents by induction on the level of situations:

1. τ is an initial situation. The B fluent restricted to the initial situations is exactly the same as the accessibility relation of M. For each unary fluent p, L interprets p at τ as M does. For each action model N and action point e of it, we let $L, \tau \models Poss(c_N(e), s)$ iff $M, \tau \models pre(e)$.
2. Assume we have interpreted *Poss* and all the fluents at level k situations. We interpret the fluents at level $k+1$ situations according to $\mathscr{D}_{ss}$. Let τ be a level $k+1$ situation. For each action model N and action point e of it, we let $L, \tau \models Poss(c_N(e), s)$ iff $M_L^\tau, \tau \models pre(e)$. Recall that the Kripke model $M_L^\tau = (S, R, V)$ is defined as follows:

   - S consists of all level $k+1$ situations of L;
   - For any $t_1, t_2 \in S$, $t_1 R_i t_2$ iff $(i, t_2, t_1) \in B^L$;
   - For each $t \in S$, $V(t)$ is the set of fluents p which are true at t in L.

By the SSA for B, it is easy to see that L satisfies F4: $B(i, s, s') \supset [Init(s) \equiv Init(s')]$. Also, L has property C1: for any situation τ, action model N and action point e of N, $L, \tau \models Poss(c_N(e), s)$ iff $M_L^\tau, \tau \models pre(e)$. So L is a model of $\mathscr{D} - \mathscr{D}_{ap}$ with C1. We now prove that $L \models \mathscr{D}_{ap}$. Since $\mathscr{R}(M_L^\tau, \tau) \cong \mathscr{R}(M_L^\tau, \tau)$, by Lemma 1, $M_L^\tau, \tau \models pre(e)$ iff $L, \tau \models \mathscr{F}(pre(e), s)$. Thus $L, \tau \models Poss(c_N(e), s)$ iff $L, \tau \models \mathscr{F}(pre(e), s)$. So $L, \tau \models Poss(c_N(e), s) \equiv \mathscr{F}(pre(e), s)$.

So we have proved that L is a model of $\mathscr{B}(\phi)$. Obviously, we have $\mathscr{R}(M, t_0) \cong \mathscr{R}(M_L^{t_0}, t_0)$. By Lemma 2, $M, t_0 \models \phi$ iff $L \models \mathscr{F}(\phi, S_0)$. Recall that (M, t_0) is a model of $\neg\phi$. So L is a model of $\mathscr{B}(\phi) \cup \{\neg\mathscr{F}(\phi, S_0)\}$. Thus $\mathscr{B}(\phi) \not\models \mathscr{F}(\phi, S_0)$. ■

6 Extended examples

In this section, we present two extended examples of modeling multi-agent scenarios in the situation calculus. In the first example, the role of each agent is not common knowledge. The second one involves both physical and sensing actions.

Example 4. Ann senses the truth value of p. Bob and Carol are observing Ann. But Ann doesn't know the role of Bob or Carol. Bob and Carol do not know the role of each other. We introduce an epistemic action $obs(e,b,c)$, which means that Ann senses the truth value of p with result e, and $b=1$ (resp. c=1) iff Bob (resp. Carol) is observing Ann. The axioms are as follows:

1. $Poss(obs(e,b,c),s) \equiv (e=1 \equiv p(s))$
2. $A(i,a,obs(e,b,c),s) \equiv$
$$[i=ann \supset (\exists b',c')a=obs(e,b',c')] \wedge$$
$$[i=bob \supset (b=0 \supset a=nil) \wedge (b=1 \supset (\exists e',c')a=obs(e',b,c'))] \wedge$$
$$[i=carol \supset (c=0 \supset a=nil) \wedge (c=1 \supset (\exists e',b')a=obs(e',b',c))]$$
3. $A(i,a,nil,s) \equiv a=nil$

The reason we have $[i=ann \supset (\exists b',c')a=obs(e,b',c')]$ is that Ann knows the sensing result but she doesn't know the role of Bob or Carol. The reason we have $[i=bob \wedge b=1 \supset (\exists e',c')a=obs(e',b,c')]$ is that Bob is observing Ann but he does not know the role of Carol.

Assume that $\mathscr{D}_{S_0}$ contains $p(S_0) \wedge \mathbf{CKnows}(\forall i \neg \mathbf{BW}(i,p),S_0)$. Then $\mathscr{D}$ entails the following, where $S_1 = do(obs(1,1,1),S_0)$.

1. $\mathbf{BW}(ann,p,S_1)$;
2. $\neg\mathbf{BW}(bob,p,S_1)$;
3. $\mathbf{Bel}(bob,\mathbf{BW}(ann,p),S_1)$;
4. $\neg\mathbf{Bel}(ann,\mathbf{Bel}(bob,\mathbf{BW}(ann,p)),S_1)$;
5. $\neg\mathbf{Bel}(carol,\mathbf{Bel}(bob,\mathbf{BW}(ann,p)),S_1)$.

Example 5. We use a simplified and adapted version of Levesque's Squirrel World. Squirrels and acorns live in a one-dimensional world unbounded on both sides. Each acorn and squirrel is located at some point, and each point can contain any number of squirrels and acorns. Acorns are completely passive. Squirrels can do the following actions:

1. $left(i)$: Squirrel i moves left a unit;
2. $right(i)$: Squirrel i moves right a unit;
3. $pick(i)$: Squirrel i picks up an acorn, which is possible when he is not holding an acorn and there is at least one acorn at his location;
4. $drop(i)$: Squirrel i drops the acorn he is holding;
5. $learn(i,n)$: Squirrel i learns that there are n acorns at his location. We use $smell(i)$ to denote $(\pi n)learn(i,n)$.

A squirrel can observe the action of another squirrel within a distance of 4, but if the action is a sensing action, the result is not observable. Initially, there are two acorns at each point. There are three squirrels: Nutty, Edgy, and Wally. Initially, they are all at point 0, holding no acorns, and have no knowledge of the number of acorns at each point, and the above is common knowledge. There are 3 ordinary fluents:

1. $hold(i,s)$: Squirrel i is holding an acorn in situation s;
2. $loc(i,p,s)$: Squirrel i is at location p in situation s;
3. $acorn(p,n,s)$: There are n acorns at location p in situation s.

For illustration, we only present some axioms of $\mathscr{D}$:

1. $Poss(pick(i),s) \equiv \neg hold(i,s) \wedge \exists p,n(loc(i,p,s) \wedge acorn(p,n,s) \wedge n > 0)$
2. $loc(i,p,do(a,s)) \equiv a = left(i) \wedge loc(i,p+1,s) \vee$
   $a = right(i) \wedge loc(i,p-1,s) \vee loc(i,p,s) \wedge a \neq left(i) \wedge a \neq right(i)$
3. $A(j,a,pick(i),s) \equiv \exists p,p'[loc(i,p,s) \wedge loc(j,p',s) \wedge$
   $(|p-p'| > 4 \supset a = nil) \wedge (|p-p'| \leq 4 \supset a = pick(i))]$
4. $A(j,a,learn(i,n),s) \equiv \exists p,p'[loc(i,p,s) \wedge loc(j,p',s) \wedge$
   $(|p-p'| > 4 \supset a = nil) \wedge (j = i \supset a = learn(i,n))$
   $(|p-p'| \leq 4 \wedge j \neq i \supset (\exists n')a = learn(i,n'))]$
5. $\mathbf{CKnows}(\forall i.loc(i,0) \wedge \neg hold(i) \wedge \forall p,n \neg \mathbf{Bel}(i,\neg acorn(p,n)),S_0)$
6. $\forall p.acorn(p,2,S_0)$

Let $\phi(s,s')$ be a formula. We introduce the following abbreviation:

$$\mathbf{Bel}(i,\phi(now,prev),s) \stackrel{def}{=} \forall s'.B(i,s',s) \supset \exists s^* \exists a^*.s' = do(a^*,s^*) \wedge \phi(s',s^*).$$

We abbreviate Nutty, Edgy, and Wally with N, E, and W, respectively. Let $\delta_1 = smell(N);pick(N)$, $\delta_2 = right(N);drop(N)$, $\delta_3 = left(W)^2;right(E)^3$, and $\delta_4 = smell(W);pick(W);left(W);left(E)$. Then $\mathscr{D}$ entails the following:

1. $Do(\delta_1,S_0,s) \supset \mathbf{TBel}(N,acorn(0,1),s) \wedge$
   $\mathbf{CKnows}(hold(N) \wedge \exists n\mathbf{TBel}(N,acorn(0,n)),s)$.
2. $Do(\delta_1;\delta_2,S_0,s) \supset \mathbf{CKnows}(\exists n(acorn(1,n,prev) \wedge acorn(1,n+1,now)),s)$.
   This says that the squirrels commonly know that there is one more acorn at point 1 now than previously.
3. $Do(\delta_1;\delta_2;\delta_3,S_0,s) \supset \mathbf{CKnows}(loc(W,-2) \wedge loc(N,1) \wedge loc(E,3),s)$.
4. $Do(\delta_1;\delta_2;\delta_3;\delta_4,S_0,s) \supset \mathbf{TBel}(N,hold(W) \wedge loc(W,-3) \wedge loc(E,2),s) \wedge$
   $\mathbf{Bel}(E,\neg hold(W) \wedge loc(W,-2),s) \wedge \mathbf{Bel}(W,loc(E,3),s)$.
   Note that now Edgy and Wally have incorrect beliefs about each other.

7 Conclusions

In this paper, by incorporating the idea of action models from DELs, we have developed a general multi-agent extension of the situation calculus. We analyzed properties of multi-agent beliefs in the situation calculus, and showed that we can provide a uniform treatment of special types of actions, such as public sensing and reading actions, noisy sensors and unintended actions, and public announcements. We showed that action model logic can be embedded into the situation calculus, and hence any multi-agent scenario which can be modeled in action model logic can be modeled in the situation calculus. Since DELs are propositional, an advantage of our work is the gain of more expressiveness and compactness in representation. We gave two extended examples to illustrate modeling of multi-agent scenarios in the situation calculus.

There are a number of topics for future research. First of all, as mentioned in the introduction, van Benthem *et al.* [8] generalized the concept of action model to that of update model which can be used to model both epistemic and physical actions. They proposed a logic, called logic of communication and change (LCC), to reason about update models. It would be interesting to explore if we can embed LCC into the situation calculus. Secondly, as shown in the Squirrel World example, because of unreliable sources of information, at certain points, agents may have incorrect beliefs about the world and other agents. When incorrect beliefs lead to inconsistent beliefs, belief revision is necessary for the agents to keep functioning in the world. The DEL community has done extensive work on multi-agent belief revision, and a good reference is [3]. The general idea is this: The semantic model is a plausibility model, where for each agent, there is a plausibility order on the set of states or actions. An agent believes ϕ if ϕ holds in the most plausible states. When we update a plausibility model by an action plausibility model, give priority to the action plausibility order. In the future, we would like to incorporate this line of work into the situation calculus. Thirdly, while the focus of the current paper is on the representation side, in the future, we would like to investigate reasoning in the multi-agent situation calculus. Finally, we would like to explore multi-agent high-level program execution and develop interesting applications of it.

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Appendix

Proposition 10. *Let* $h : \mathscr{R}(M_1,t_1) \cong \mathscr{R}(M_2,t_2)$. *Then for any AML formula* ϕ, $M_1,t_1 \models \phi$ *iff* $M_2,t_2 \models \phi$.

Proof. We prove by induction on the complexity of ϕ. The cases that ϕ is p, $\neg\psi$, or $\psi \wedge \psi'$ are easy. We prove the remaining cases:

1. ϕ is $B_i\psi$. Let t be a state of M such that t_1R_it. By Proposition 9, $\mathscr{R}(M_1,t) \cong \mathscr{R}(M_2,h(t))$. By induction hypothesis, $M_1,t \models \psi$ iff $M_2,h(t) \models \psi$. So $M_1,t_1 \models B_i\psi$ iff for every t such that t_1R_it, $M_1,t \models \psi$ iff for every t' such that t_2R_it', $M_2,t' \models \psi$ iff $M_2,t_2 \models B_i\psi$.
2. ϕ is $C_{\mathscr{E}}\psi$. Similar to the above case.
3. ϕ is $[N,e_0]\psi$. Let t be a state of M reachable from t_1. By Proposition 9, $\mathscr{R}(M_1,t) \cong \mathscr{R}(M_2,h(t))$. Let e be an action point of N. By induction hypothesis, $M_1,t \models pre(e)$ iff $M_2,h(t) \models pre(e)$. Now suppose $M_1,t_1 \models pre(e_0)$.

We show that $\mathscr{R}((M_1,t_1)\otimes(N,e_0)) \cong \mathscr{R}((M_2,t_2)\otimes(N,e_0))$. Let (t,e) be a state of $\mathscr{R}((M_1,t_1)\otimes(N,e_0))$. Then $M_1,t \models pre(e)$ and (t,e) is reachable from (t_1,e_0). So $M_2,h(t) \models pre(e)$, and $(h(t),e)$ is reachable from (t_2,e_0) (since $t_2 = h(t_1)$ and h preserves the accessibility relation). Hence $(h(t),e)$ is a state of $\mathscr{R}((M_2,t_2)\otimes(N,e_0))$. Let $g((t,e)) = (h(t),e)$. It is easy to show that g is a bijection from the states of $\mathscr{R}((M_1,t_1)\otimes(N,e_0))$ to those of $\mathscr{R}((M_2,t_2)\otimes(N,e_0))$, g preserves the accessibility relation and the atoms. So if $M_1,t_1 \models pre(e_0)$, then $\mathscr{R}((M_1,t_1)\otimes(N,e_0)) \cong \mathscr{R}((M_2,t_2)\otimes(N,e_0))$. By induction hypothesis, $(M_1,t_1)\otimes(N,e_0) \models \psi$ iff $(M_2,t_2)\otimes(N,e_0) \models \psi$. Thus $M_1,t_1 \models [N,e_0]\psi$ iff if $M_1,t_1 \models pre(e_0)$ then $(M_1,t_1)\otimes(N,e_0) \models \psi$ iff if $M_2,t_2 \models pre(e_0)$ then $(M_2,t_2)\otimes(N,e_0) \models \psi$ iff $M_2,t_2 \models [N,e_0]\psi$. ■

Proposition 11. *Let L be a model of $\mathscr{D} - \mathscr{D}_{ap}$ with C1, and τ_0 a situation of L. Let (M,t_0) be a pointed Kripke model, and (N,e_0) a pointed action model. Suppose that $h : \mathscr{R}(M,t_0) \cong \mathscr{R}(M_L^{\tau_0},\tau_0)$ and $M,t_0 \models pre(e_0)$. Then $\mathscr{R}((M,t_0)\otimes(N,e_0)) \cong \mathscr{R}(M_L^{\tau_1},\tau_1)$ where τ_1 is $do^L(c_N(e_0)^L,\tau_0)$.*

Proof. To begin with, we show that for any state t of M reachable from t_0 and for any action point e of N, $M,t \models pre(e)$ iff $L,h(t) \models Poss(c_N(e),s)$. Since $h : \mathscr{R}(M,t_0) \cong \mathscr{R}(M_L^{\tau_0},\tau_0)$, by Proposition 9, $\mathscr{R}(M,t) \cong \mathscr{R}(M_L^{\tau_0},h(t))$. By Proposition 10, $M,t \models pre(e)$ iff $M_L^{\tau_0},h(t) \models pre(e)$. By C1, $L,h(t) \models Poss(c_N(e),s)$ iff $M_L^{\tau_0},h(t) \models pre(e)$. Thus $M,t \models pre(e)$ iff $L,h(t) \models Poss(c_N(e),s)$.

We define a function g from the states of $\mathscr{R}((M,t_0)\otimes(N,e_0))$ to the situations of L as follows: $g((t,e)) = do^L(c_N(e)^L,h(t))$. We first show that g is an injection. Suppose that $do^L(c_N(e_1)^L,h(t_1)) = do^L(c_N(e_2)^L,h(t_2))$. Then by F1, $c_N(e_1)^L = c_N(e_2)^L$ and $h(t_1) = h(t_2)$. By $\mathscr{D}_{una}$, $e_1^L = e_2^L$. By A0, $e_1 = e_2$. Since h is an injection, $t_1 = t_2$.

We now show that g preserves the accessibility relation. Let (t_1,e_1) and (t_2,e_2) be two states of $\mathscr{R}((M,t_0)\otimes(N,e_0))$. Then $M,t_i \models pre(e_i)$, $i = 1,2$. So $L,h(t_i) \models Poss(c_N(e_i),s)$. Thus $(t_1,e_1)\,R_i'\,(t_2,e_2)$ iff $t_1\,R_i\,t_2$ and $e_1 \to_i e_2$ iff $(i,h(t_2),h(t_1)) \in B^L$ and $L,h(t_1) \models A(i,c_N(e_2),c_N(e_1),s)$ (by A3) iff $(i,g((t_2,e_2)),g((t_1,e_1))) \in B^L$ (by the SSA for B). Next, we show that g preserves the fluents. For any state (t,e) of $\mathscr{R}((M,t_0)\otimes(N,e_0))$, for any fluent p, p is true at (t,e) iff p is true at t iff p is true at $h(t)$ iff p is true at $do^L(c_N(e)^L,h(t))$, which is $g((t,e))$, by the SSA for p.

We now show that g is a function from the states of $\mathscr{R}((M,t_0)\otimes(N,e_0))$ to those of $\mathscr{R}(M_L^{\tau_1},\tau_1)$. Let (t,e) be a state of $\mathscr{R}((M,t_0)\otimes(N,e_0))$. Then (t,e) is reachable from (t_0,e_0). Since g preserves the accessibility relation, $g(t,e)$ is reachable from $g(t_0,e_0)$, which is τ_1. Thus $g(t,e)$ is a state of $\mathscr{R}(M_L^{\tau_1},\tau_1)$.

Finally, we show that g is a surjection. Let ω_1 be a situation of L that is reachable from τ_1 by a B-path. Since $M,t_0 \models pre(e_0)$, $L,\tau_0 \models Poss(c_N(e_0),s)$. By A3 and the SSA for the B fluent, there exist an action point e of N reachable from e_0 and a situation ω_0 of L reachable from τ_0, such that $\omega_1 = do^L(c_N(e)^L,\omega_0)$ and $L,\omega_0 \models Poss(c_N(e),s)$. Since $h : \mathscr{R}(M,t_0) \cong \mathscr{R}(M_L^{\tau_0},\tau_0)$, there exists a state t of M reachable from t_0 such that $h(t) = \omega_0$. Thus $M,t \models pre(e)$, (t,e) is reachable from (t_0,e_0), and $\omega_1 = g((t,e))$. ■

References

1. F. Bacchus, J. Halpern, and H. Levesque. Reasoning about noisy sensors in the situation calculus. *Artificial Intelligence*, 111:171–208, 1999.
2. A. Baltag and L. Moss. Logics for epistemic programs. *Synthese*, 139(2):165–224, 2004.
3. A. Baltag and S. Smets. A qualitative theory of dynamic interactive belief revision. In *Texts in Logic and Games, Vol 3*. 2008.
4. A. Baltag, L. S. Moss, and S. Solecki. The logic of public announcements, common knowlegde, and private suspicions. In *Proc. of the Conference on Theoretical Aspects of Rationality and Knowledge (TARK-98)*, 1998.
5. C. Baral. Reasoing about actions and change: from single agent actions to multi-agent actions. In *Proc. of the International Conference on Principles of Knowledge Representation and Reasoning (KR-10)*, 2010.
6. V. Belle and G. Lakemeyer. Reasoning about imperfect information games in the epistemic situation calculus. In *Proc. of the AAAI Conference on Artificial Intelligence (AAAI-10)*, 2010.
7. J. van Benthem. McCarthy variations in a modal key. *Artificial Intelligence*, 175(1):428–439, 2011.
8. J. van Benthem, J. van Eijck, and B. Kooi. Logics of communication and change. *Information and Computation*, 204(11):1620–1662, 2006.
9. J. P. Delgrande and H. J. Levesque. Belief revision with sensing and fallibe actions. In *Proc. of the International Conference on Principles of Knowledge Representation and Reasoning (KR-12)*, 2012.
10. H. van Ditmarsch, W. van der Hoek, and B. Kooi. *Dynamic Epistemic Logic*. Springer, 2007.
11. H. van Ditmarsch, A. Herzig, and T. de Lima. From situation calculus to dynamic epistemic logic. *J. of Logic and Computation*, 21(2):179–204, 2011.
12. R. F. Kelly and A. R. Pearce. Complex epistemic modalities in the situation calculus. In *Proc. of the International Conference on Principles of Knowledge Representation and Reasoning (KR-08)*, 2008.
13. G. Lakemeyer and H. J. Levesque. Situations, si! Situation terms, no! In *Proc. of the International Conference on Principles of Knowledge Representation and Reasoning (KR-04)*, 2004.
14. G. Lakemeyer and H. J. Levesque. Semantics for a useful fragment of the situation calculus. In *Proc. of the International Joint Conference on Artificial Intelligence (IJCAI-05)*, 2005.
15. H. J. Levesque, R. Reiter, Y. Lespérance, F. Lin, and R. B. Scherl. GOLOG: A logic programming language for dynamic domains. *J. Logic Programming*, 31(1-3), 1997.
16. J. McCarthy and P. J. Hayes. Some philosophical problems from the standpoint of artificial intelligence. In *Machine Intelligence 4*, pages 463–502. 1969.
17. J. Plaza. Logics of public communications. In *Proc. of the Fourth International Symposium on Methodologies for Intelligent Systems (ISMIS-89)*, 1989.
18. R. Reiter. The frame problem in the situation calculus: A simple solution (sometimes) and a completeness result for goal regression. In *Artificial Intelligence and Mathematical Theory of Computation: Papers in Honor of John McCarthy*, pages 359–380. 1991.
19. R. Reiter. *Knowledge in Action: Logical Foundations for Specifying and Implementing Dynamical Systems*. MIT Press, 2001.
20. R. B. Scherl and H. J. Levesque. The frame problem and knowledge-producing actions. In *Proc. of the AAAI Conference on Artificial Intelligence (AAAI-93)*, 1993.
21. R. B. Scherl and H. J. Levesque. Knowledge, action, and the frame problem. *Artificial Intelligence*, 144(1-2):1–39, 2003.
22. S. Shapiro, Y. Lespérance, and H. J. Levesque. Specifying communicative multi-agent systems. In *Agents and Multi-Agent Systems – Formalisms, Methodologies, and Applications*, volume 1441 of *Lecture Notes in Computer Science*, pages 1–14. Springer, 1998.